F500: Empirical Finance
Lecture 9: Intertemporal Equilibrium Pricing

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Outline

1. The Stochastic Discount Factor
2. The Consumption Capital Asset Pricing Model
3. The Equity Premium Puzzle
4. Explanations for the Puzzle
5. Other approaches

Reading: Linton (2019), Chapter 10.

Intertemporal Optimization

Lucas (1978). Investors have a concave, positively sloped, time-invariant utility function for consumption, and a constant rate of time preference δ. They invest in risky assets $\{R_{i,t}\}_{i=1}^{n}$ and consume the proceeds over time $\{C_t\}$

Investors choose investment/consumption to maximize the discounted expected utility of lifetime consumption

$$V_t = E_t\left[\sum_{j=0}^{\infty}\delta^j U\left(C_{t+j}\right)\right] = U\left(C_t\right) + \delta \overbrace{E_t V_{t+1}}^{cont.\ value}$$

subject to a budget constraint

$$W_{t+1} = \left(W_t - C_t\right)\sum_{i=1}^{N} w_{it}\left(1 + R_{i,t+1}\right)$$

First order condition for each risky asset i, the so-called Euler equation

$$U'(C_t) = \delta E_t \left[(1 + R_{i,t+1}) \, U'(C_{t+1}) \right],$$

where E_t means expectat conditional on information at time t.
Defining

$$M_{t+1} = \delta \frac{U'(C_{t+1})}{U'(C_t)}$$

and rearranging the series of first-order conditions $(i = 1, \ldots, n)$

$$1 = E_t \left[M_{t+1} (1 + R_{i,t+1}) \right]$$

Stochastic Discount Factor

The random variable $M_t > 0$ is called the stochastic discount factor or pricing kernel. It is the (random) ratio of marginal utilities between each "investment" date-state and "realized return" date-state, weighted by pure time preference.

- Pricing formula for any asset

$$P_t = E_t\left[M_{t+1}X_{t+1}\right],$$

  where X_{t+1} is the cash flow in period $t+1$ (e.g., $P_{t+1} + D_{t+1}$)
- Relationship can be derived more generally from non-arbitrage assumption: There does not exist a negative-cost portfolio with a uniformly non-negative payoff.

Risk Neutral Expectation

Replace the true probability weights (denoted $\mathcal{P}$) in the basic pricing expectation with "hypothetical" probability weights

$$\mathcal{P}^* \propto M_t \times \mathcal{P},$$

and taking expectations under these transformed probabilities gives

$$P_t = E_t[M_{t+1}X_{t+1}] = E_t^*[X_{t+1}]$$

All assets have the same expected return under the transformed probabilities.

This new hypothetical "probability" measure is called the equivalent martingale measure or the risk neutral measure. It is very useful for empirical derivatives pricing (not covered in this course).

The Consumption Capital Asset Pricing Model Again

Adding and subtracting $(M_{t+1} = E_t(M_{t+1}) + M_{t+1} - E_t(M_{t+1}))$, we obtain

$$\begin{aligned} 1 &= E_t\left[(1+R_{i,t+1})\, M_{t+1}\right] \\ &= E_t\left[(1+R_{i,t+1})\, E_t(M_{t+1})\right] + E_t\left[(1+R_{i,t+1})\,(M_{t+1} - E_t(M_{t+1}))\right] \\ &= E_t\left[(1+R_{i,t+1})\right] E_t[M_{t+1}] + \mathrm{cov}_t\left(R_{i,t+1}, M_{t+1}\right) \end{aligned}$$

Let R_{0t} denote an asset such that

$$\mathrm{cov}_t(R_{0,t+1}, M_{t+1}) = 0$$

(zero beta or risk free asset). Then

$$E_t[M_{t+1}] = \frac{1}{E_t\left[1 + R_{0,t+1}\right]} = \delta E_t\left[\frac{U'(C_{t+1})}{U'(C_t)}\right]$$

Then substituting in

$$1 = E_t\left[(1 + R_{i,t+1})\right]\frac{1}{E_t\left[1 + R_{0,t+1}\right]} + \mathrm{cov}_t\left(R_{i,t+1}, M_{t+1}\right)$$

and rearranging we obtain for any asset i

$$\begin{aligned} E_t[R_{i,t+1} - R_{0,t+1}] &= -\mathrm{cov}_t(R_{i,t+1}, M_{t+1}) \times E_t\left(1 + R_{0,t+1}\right) \\ &= -\mathrm{cov}_t\left(R_{i,t+1}, \delta\frac{u'(C_{t+1})}{u'(C_t)}\right) \times \frac{1}{\delta E_t\left[\frac{U'(C_{t+1})}{U'(C_t)}\right]} \end{aligned}$$

- An asset whose covariance with M_t is negative tends to have low returns when the investor's marginal utility of consumption is high ie when consumption is low. Require a large risk premium to hold it.

- Suppose there is an asset R_{mt} that pays off exactly M_t then

$$E_t[R_{m,t+1} - R_{0,t+1}] = -\mathrm{var}_t(M_{t+1}) \times E_t\left(1 + R_{0,t+1}\right)$$

Therefore,

$$E_t[R_{i,t+1} - R_{0,t+1}] = \beta_{im,t} E_t[R_{m,t+1} - R_{0,t+1}]$$

$$\beta_{im,t} = \frac{\mathrm{cov}_t(R_{i,t+1}, R_{m,t+1})}{\mathrm{var}_t(R_{m,t+1})}$$

This pricing model is called the consumption CAPM.

- We can also, starting from $1 = E\left[\left(1 + R_{i,t+1}\right) M_{t+1}\right]$, derive an unconditional version

$$E[R_{it} - R_{0t}] = \beta_{im} E[R_{mt} - R_{0t}], \quad \beta_{im} = \frac{\mathrm{cov}(R_{it}, R_{mt})}{\mathrm{var}(R_{mt})}$$

Note that $\beta_{im} \neq E\beta_{im,s}$.

Note that the CCAPM model has:

- Cross-sectional predictions (relative risk premia are proportional to consumption betas),
- Time-series predictions (expected returns vary with expected consumption growth rates, etc.),
- Joint time-series/cross-sectional predictions.

The standard CAPM only has cross-sectional predictions.

Econometric Testing

Need to specify $U(.)$ in order to estimate betas from consumption data. Simple elegant utility function is the CRRA class with risk aversion parameter γ

$$U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

Calculating the stochastic discount factor gives

$$M_{t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

$$m_{t+1} = \log M_{t+1} = \log \delta - \gamma g_{t+1} \quad ; \quad g_{t+1} = \log (C_{t+1} / C_t)$$

The "riskless" asset satisfies

$$1 + R_{ft} = \frac{1}{\delta} E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{\gamma} \right]$$

How to test the consumption CAPM?

- Hansen and Singleton (1982) GMM conditional moment restriction

$$E_t\left[(1+R_{it+1})\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}-1\right]=0$$

- Do not need to specify dynamics for returns or consumption except stationarity on consumption growth.
- Convert to unconditional moment restriction and do GMM

- Let with X_t denoting all the data and $\theta = (\delta, \gamma)$

$$g(X_t, \theta) = \left[\begin{pmatrix} 1 + R_{1,t+1} \\ \vdots \\ 1 + R_{n,t+1} \end{pmatrix} \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} - 1 \right] \otimes \overbrace{Z_t}^{\text{instruments}} \quad \in \mathbb{R}^p$$

- Then we have the unconditional moment restriction

$$E\left[g(X_t, \theta)\right] = 0.$$

- Estimate the parameters θ by the Generalized Method of Moments (GMM) using $p > 2$ sample moments and quadratic form

$$G_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} g(X_t, \theta) \quad ; \quad \min_{\theta} G_T(\theta)^\intercal W G_T(\theta)$$

This is nonlinear in θ.

Test whether overidentifying restrictions ($p > 2$) hold using the J-test.

$$||G_T(\widehat{\theta})||_{W_{opt}} = G_T(\widehat{\theta})^\intercal W_{opt} G_T(\widehat{\theta})$$

This is asymptotically chi squared (χ^2_{p-2}) under the null hypothesis that the moments are correct.

- Empirically the CCAPM model performs very poorly, see below. The empirical failure of the consumption CAPM is among the most important anomalies of asset pricing theory.

The Equity Premium Puzzle

- We make an assumption that log consumption growth and log equity market return are jointly normal (can hold a little more generally in an approximate sense like the Campbell log linearization) and that utility is CRRA. We have with r_{it} the logarithmic returns and $g_{t+1} = \log(C_{t+1}/C_t)$

$$\begin{aligned} & \log E_t \left[(1+R_{it+1})\delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] \\ = \; & E_t \log \left[(1+R_{it+1})\delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] \\ & + \frac{1}{2} \mathrm{var}_t \log \left[(1+R_{it+1})\delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] \end{aligned}$$

$$\begin{aligned} = \; & E_t r_{i,t+1} + \log \delta - \gamma E_t g_{t+1} \\ & + \frac{1}{2} \left[\mathrm{var}_t r_{i,t+1} + \gamma^2 \mathrm{var}_t g_{t+1} - 2\gamma \mathrm{cov}_t(g_{t+1}, r_{i,t+1}) \right] \end{aligned}$$

- Then we have the linearish (in parameters) equation

$$0 = E_t\left[r_{i,t+1}\right] + \log\delta - \gamma E_t\left[g_{t+1}\right] + \frac{1}{2}\left[\sigma_i^2(t) + \gamma^2\sigma_c^2(t) - 2\gamma\sigma_{ic}(t)\right],$$

where

$$\sigma_{ic}(t) = \mathrm{cov}_t(r_{i,t+1}, g_{t+1})$$

$$\sigma_i^2(t) = \mathrm{var}_t(r_{i,t+1})$$

$$\sigma_c^2(t) = \mathrm{var}_t(g_{t+1})$$

- If we assume conditional homoskedasticity we can use this to obtain estimating equations or just provide interpretation

- The risk-free rate is determined endogenously in the model (setting $\sigma_i^2 = \sigma_{ic} = 0$)

$$E[r_{ft}] = -\log\delta + \gamma g - \frac{\gamma^2\sigma_c^2}{2}$$

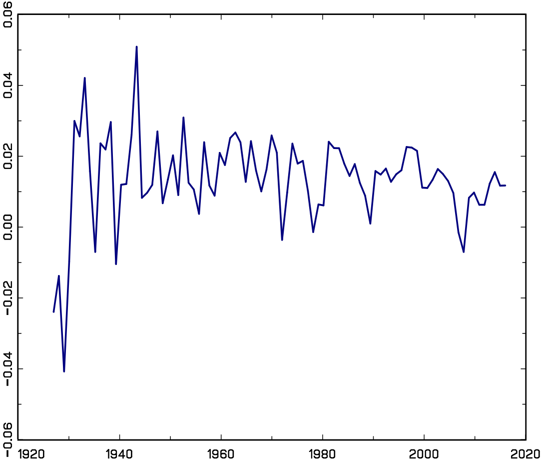
where g is the mean growth rate of consumption. Risk free rate depends on impatience, risk aversion, consumption growth and volatility.

- For any other asset i we have

$$E[r_{i,t+1} - r_{f,t+1}] = \gamma\sigma_{ic} - \frac{\sigma_i^2}{2} \leq \gamma\sigma_{ic}$$

This is a pricing equation for the risk premium in terms of covariation with consumption growth.

Consumption is not very variable. The growth of real annual per capita expenditure variable (rPCEa) is shown below. Its mean is $\overline{g} = 0.0134$ and standard deviation $s_g = 0.0127$, which is much less than the variation of stock returns



The Equity Premium Puzzle. Empirically, σ_{ic} is very small relative to the observed premium of equities over fixed income securities, hence this implies a very high coefficient of risk aversion γ.

The Risk Free Rate Puzzle. If γ is set high enough to explain observed equity risk premia, it is too high (given average consumption growth) to explain observed risk-free returns! The rate of pure time preference is driven below zero.

Mehra and Prescott (1985).

"Historically the average return on equity has far exceeded the average return on short-term virtually default-free debt. Over the ninety-year period 1889-1978 the average real annual yield on the Standard and Poor 500 Index was seven percent, while the average yield on short-term debt was less than one percent. The question addressed in this paper is whether this large differential in average yields can be accounted for by models that abstract from transactions costs, liquidity constraints and other frictions absent in the Arrow-Debreu set-up. Our finding is that it cannot be, at least not for the class of economies considered. "

Explanations for the Equity Premium Puzzle

A large number of explanations for the puzzle have been proposed. These include:

- a contention that the equity premium does not exist: that the puzzle is a statistical illusion
- modifications to the assumed preferences of investors, and
- imperfections in the model of risk aversion.

Statistical Illusion

The most basic explanation is that there is no puzzle to explain: that there is no equity premium. Essentially, we don't have enough statistical power to distinguish the equity premium from zero.

Sample selection bias: US equity market is the most intensively studies in equity market research. Not coincidentally, it had the best equity market performance in the 20th century; others (e.g. Russia, Germany, and China) produced a gross return of zero due to bankruptcy events.

Low number of data points: the period 1900–2005 provides only 105 independent years which is not a large number of years statistically.

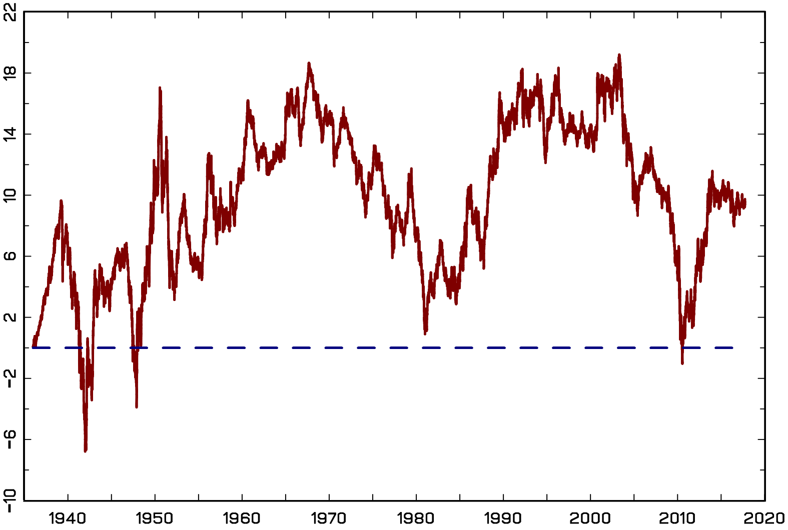
Sample period choice: returns of equities (and relative returns) vary greatly depending on which points are included. Using data starting from the top of the market in 1929 or starting from the bottom of the market in 1932 (leading to estimates of equity premium of 1% lower per year), or ending at the top in 2000 (vs. bottom in 2002) or top in 2007 (vs. bottom in 2009 or beyond) completely change the overall conclusion.

Is there an equity premium puzzle in the USA?

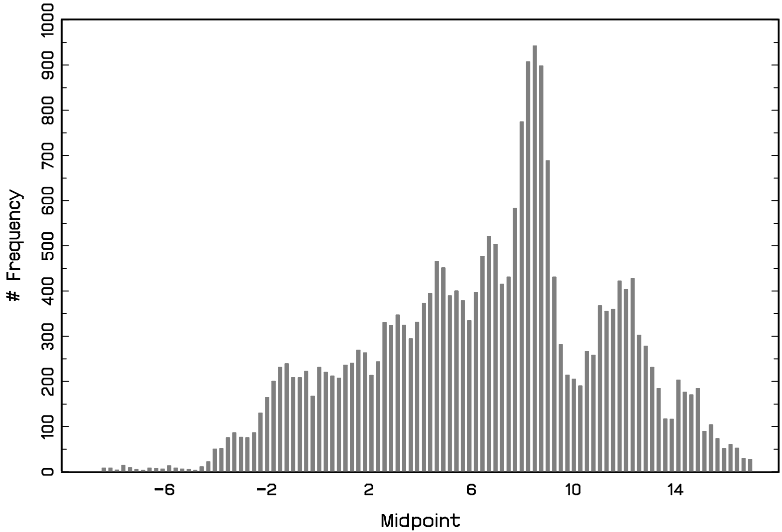
We report the estimated market risk premium using the FF market factors, the annualized daily return series and the annual return series. For the daily return series there are $n = 24034$ observations for which $1/\sqrt{n} = 0.00645$ and for the annual return series $n = 90$ for which $1/\sqrt{n} = 0.1054$.

	μ	med	σ	$IQR/1.349$	$\rho(1)$
(1926-2016) Annualized Daily excess returns	7.320	15.120	16.906	10.473	0.0679
(1926-2016) Annual excess returns	8.48	10.735	20.29	20.167	0.0214

Table: Market risk premium



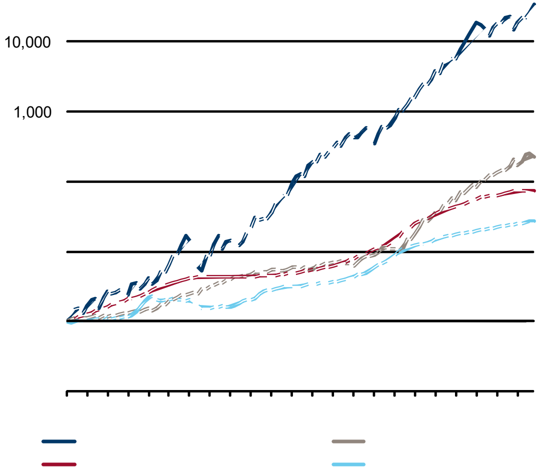
Rolling window trailing 10 year gross nominal returns on the CRSP value weighted index



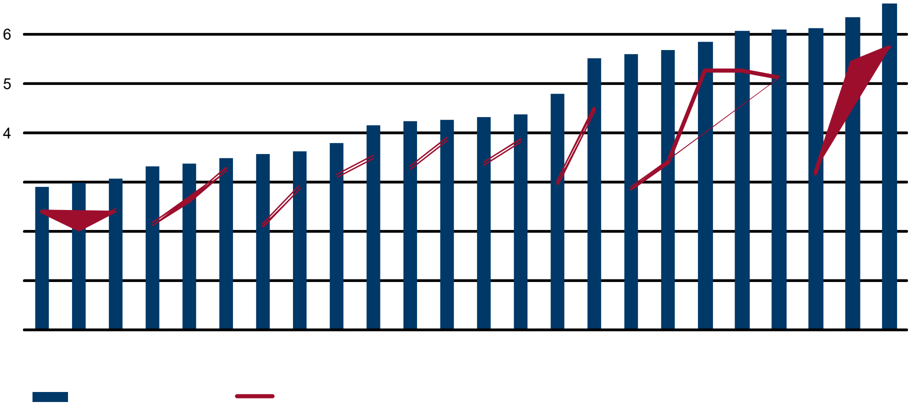
Distribution of the Annual Risk Premium on the FF Market factor from Ten years of Daily data

- First, there is considerable variation in long horizon returns around the very long run average of around 10% per year.
- Second the series itself is quite predictable, a predictability that has been manufactured out of the rolling window construction.

Nominal Returns in the US since 1900 (Dimson, Marsh, and Staunton)



Around the World



Market frictions

- If some or all assets cannot be sold short by some or all investors, then the stochastic discount factor equation is much weaker:

$$E\left[\left(1+R_{it}\right)M_t\right] \leq 1$$

- The inequality-version of the stochastic discount factor does not aggregate across investors. Hence aggregate consumption is not directly relevant.

Separating Risk Aversion and Intertemporal Substitution

- The standard multiperiod von Neumann-Morgenstern utility function is elegant but may not provide an accurate representation.
- Multiperiod von Neumann-Morgenstern utility has a single parameter (the risk aversion parameter) that governs both the elasticity of intertemporal substitution and Arrow-Pratt risk aversion.

Elasticity of intertemporal substitution

Consider an investor who consumes/saves in period zero and consumes in period one with no risk. Suppose that the risk-free interest rate is R_f. The elasticity of intertemporal substitution is defined as the percentage change in optimal consumption growth for a percentage change in the risk-free interest rate:

$$EIS = \frac{\%\partial(C_1 / C_0)}{\%\partial(R_f)}$$

In the CRRA case it is easy to show

$$EIS = \frac{1}{\gamma}.$$

Note that EIS is an intertemporal concept with no connection to risk.

Arrow-Pratt Risk Aversion

Consider the family of risky investments $x(\delta) = \delta\pi + \sqrt{\delta}z$, where π is a constant and z is a unit-variance, zero-mean random variable. Note that for any $\delta > 0$ the mean/variance ratio of this investment equals π. Let C denote a riskless consumption level and $U(C)$ a vN-M utility function. The absolute risk aversion of $U(C)$ is the value of π which leaves the investor approximately indifferent for small values of δ

$$ARA = \{\pi \ s.t. \ \lim_{\delta \to 0} E[U(C + x(\delta))] = U(C)\}$$

It is easy to show that

$$ARA = -\frac{U''(C)}{U'(C)}.$$

It is not difficult to show that if ARA is constant for all C then $U(C) = \exp(-\gamma C)$ for some γ.

The relative risk aversion is the ARA divided by the level of consumption:

$$RRA = ARA/C$$

It is not difficult to show that if RRA is constant for all C then (choosing a convenient scaling for the utility function) $U(C) = \frac{1}{1-\gamma}C^{1-\gamma}$ which is the CRRA utility function.
Given CRRA then RRA $= \gamma$ for all C.

Note that RRA is a pure risk concept with no intertemporal component whereas EIR is a pure intertemporal concept with no risk component. In the multiperiod vN-M framework they are inextricably linked together.

The Epstein-Zin-Weil "utility" function

- Separates EIS and RRA. Non EU preferences defined recursively by

$$U_t = \left\{ (1-\delta) C_t^{\frac{1-\gamma}{\theta}} + \delta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}$$

  where δ is discount factor, γ is coefficient of relative risk aversion ψ is the elasticity of intertemporal substitution and $\theta = (1-\gamma)/(1-\frac{1}{\psi}) :$ $\gamma > 1/\psi$ the agent prefers early resolution of uncertainty

- The first-order conditions are more complex than in the vN-M case, but one useful series of first-order conditions is

$$E_t \left[\left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right\}^{\theta} \left\{ \frac{1}{(1+R_{m,t+1})} \right\}^{1-\theta} (1+R_{i,t+1}) \right] = 1$$

where $R_{m,t}$ is the return on the market portfolio.

- Assuming that consumption and the return on the market portfolio are jointly lognormal and conditionally homoskedastic, and substituting gives

$$r_{f,t+1} = -\log\delta + \frac{\theta-1}{2}\sigma_m^2 - \frac{\theta}{2\psi^2}\sigma_c^2 + \frac{1}{\psi}E_t\left[g_{t+1}\right]$$

$$E_t\left[r_{i,t+1} - r_{f,t+1}\right] = \frac{\theta}{\psi}\overbrace{\sigma_{ic}}^{\text{consumption beta}} + (1-\theta)\overbrace{\sigma_{im}}^{\text{market beta}} - \frac{\sigma_i^2}{2}$$

  This says that consumption betas and market portfolio betas both affect asset risk premia.
- With consumption data one can test this model; performs better than standard utility model
- Aggregate consumption data only available quarterly and not well measured

Eliminate Consumption

- Campbell (1993) shows how to eliminate consumption. Applying Campbell's log-linearisation from a previous lecture one can obtain

$$E_t\left[r_{i,t+1} - r_{f,t+1}\right] = \gamma\sigma_{im} + (\gamma - 1)\,\sigma_{ih} - \frac{\sigma_i^2}{2}$$

$$\sigma_{ih} = \mathrm{cov}_t\left(r_{i,t+1}, \sum_{j=1}^{\infty}\rho^j\left\{E_{t+1}r_{m,t+1+j} - E_t r_{m,t+1+j}\right\}\right)$$

- Risk premia depend on market betas and on "changing opportunity set betas", σ_{ih}. Covariation with news about future returns to the market affects risk premia.
- The EIS parameter θ/ψ is also eliminated.
- However, need to specify a model to calculate $E_{t+1}r_{m,t+1+j}$

Vector Autoregressions

To bring this to data we make some strong asumptions, specifically that the relevant data are generated by a VAR process. For example suppose that

$$X_t = (r_{mt}, yield_t, etc., ...)$$

where the first element is the market return and the other are observable state variables interacting with consumption

Definition

Suppose that

$$X_{t+1} = AX_t + \varepsilon_{t+1},$$

$$\begin{bmatrix} X_{1t+1} \\ \vdots \\ X_{Kt+1} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1K} \\ \vdots & & \\ a_{K1} & & a_{KK} \end{bmatrix} \begin{bmatrix} X_{1t} \\ \vdots \\ X_{Kt} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t+1} \\ \vdots \\ \varepsilon_{Kt+1} \end{bmatrix}$$

where $A = (a_{ij})$ is a parameter matrix and ε_{t+1} is an error vector i.i.d mean zero.

This allows us to measure σ_{ih} or rather the expectation term inside the covariance. Let $e_1^\intercal = (1, 0, \ldots, 0)$, then

$$r_{m,t+1} = e_1^\intercal X_{t+1} = e_1^\intercal AX_t + e_1^\intercal \varepsilon_{t+1}$$

We can forecast the future of X_t by

$$E_t X_{t+1} = AX_t, \quad E_t X_{t+j} = A^j X_t$$

Therefore, in particular

$$E_t r_{m,t+1} = e_1^\intercal AX_t, \quad E_t r_{m,t+j} = e_1^\intercal A^j X_t$$

We apply this to the terms inside σ_{ih} to obtain

$$
\begin{aligned}
& E_{t+1}\left[\sum_{j=1}^{\infty}\rho^{j} r_{m,t+1+j}\right] - E_t\left[\sum_{j=1}^{\infty}\rho^{j} r_{m,t+1+j}\right] \\
= & \sum_{j=1}^{\infty}\rho^{j} e_1^{\intercal} A^{j} X_{t+1} - \sum_{j=1}^{\infty}\rho^{j} e_1^{\intercal} A^{j+1} X_t \\
= & e_1^{\intercal}\sum_{j=1}^{\infty}\rho^{j} A^{j}\varepsilon_{t+1} \\
= & e_1^{\intercal}\rho A\left(1-\rho A\right)^{-1}\varepsilon_{t+1} \equiv \varphi^{\intercal}\varepsilon_{t+1} = \sum_{k=1}^{K}\varphi_k\varepsilon_{k,t+1}
\end{aligned}
$$

- The factor betas φ are nonlinear combinations of the VAR coefficients and the extra-market factors are the VAR innovations.
- Let

$$\sigma_{ik} = \mathrm{cov}(r_{i,t+1}, \varepsilon_{k,t+1})$$

- Inserting this into the above equation gives

$$E_t\left[r_{i,t+1} - r_{f,t+1}\right] = -\frac{\sigma_i^2}{2} + \gamma\sigma_{i1} + (\gamma - 1)\sum_{k=1}^{K} \varphi_k \sigma_{ik}$$

which (except for the log expectation adjustment) is identical to the multi-beta pricing models tested previously.

- Campbell (1996) estimates using annual data postwar.
  - He finds that $\varphi_1 << 0$ (that is, the correlation between market return innovations and revisions in expected future market returns is negative).
  - Other coefficients φ_k are not significant.

Other Asset Pricing Approaches

Habit models

- Difference model (Constantinides (1990))

$$U_t = E_t \sum_{j=0}^{\infty} \delta^j \frac{\left(C_{t+j} - X_{t+j}\right)^{1-\gamma} - 1}{1-\gamma}$$

- Ratio model (Abel (1990))

$$U_t = E_t \sum_{j=0}^{\infty} \delta^j \frac{\left(C_{t+j} / X_{t+j}\right)^{1-\gamma} - 1}{1-\gamma}$$

  Habit X_t, for example X_t some level of previous consumption. Gives additional flexibility, but not very plausible.

- Hyperbolic discounting (Laibson (1996))

$$U(C_t) + \beta E_t \sum_{j=0}^{\infty} \delta^j U\left(C_{t+j}\right)$$

Lettau and Ludvigson (2001,2004)

- Standard dynamic optimization with wealth W. LL assume that wealth is composed of asset holdings A and human capital H and that H is related to labor income Y in a specific way
- Let r_w be the log of net return on aggregate wealth. By linearization and solving forward they obtain

$$c_t - w_t = \sum_{i=1}^{\infty} \rho_w^i E_t \left(r_{w,t+i} - \Delta c_{t+i} \right)$$

- Approximating the nonstationary component of human capital by aggregate labour income, they obtain

$$\overbrace{c_t - \alpha_a a_t - \alpha_y y_t}^{cay_t} = \sum_{i=1}^{\infty} \rho_w^i E_t \left((1-\nu) r_{a,t+i} + \nu \Delta y_{t+i} - \Delta c_{t+i} \right)$$

the silver bullet. Data available at https://sites.google.com/view/martinlettau/data

- The prediction is that consumption, asset values and income are cointegrated. Their residual summarizes the expectations of future returns on the market portfolio.
- Using U.S. quarterly stock market data, they find that fluctuations in the consumption–wealth ratio (cay) are strong predictors of both real stock returns and excess returns over a Treasury bill rate.
- They find that this variable is a better forecaster of future returns at short and intermediate horizons than is the dividend yield, the dividend payout ratio, and several other popular forecasting variables.

Long run risks model Bansal and Yaron (2004)

- Epstein Zin preferences

$$E_t \left[\left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right\}^{\theta} \left\{ \frac{1}{(1+R_{c,t+1})} \right\}^{1-\theta} (1+R_{i,t+1}) \right] = 1$$

  where R_c is the gross return on an asset that delivers aggregate consumption as its dividend each period (like, but not equal to, the market portfolio).

- In logs with $g_{t+1} = \log C_{t+1}/C_t$ and $z_t = \log(P_t/C_t)$, where P is the price level

$$r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_t$$

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{c,t+1}$$

They specify dynamics for consumption and dividend growth rates

$$g_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$$

$$g_{d,t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}$$

where the unobserved state variables (x is the "Long Run Risks") are

$$x_{t+1} = \rho x_t + \varphi_e \sigma_t e_{t+1}$$

$$\sigma_{t+1}^2 = \sigma^2 + \nu_1 (\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}$$

with innovations $e_{t+1}, w_{t+1}, \eta_{t+1},\ u_{t+1}$ are standard normal and iid and mutually independent. The parameter $\rho \rightsquigarrow 1$ reflects persistence of growth process.

- Captures the idea that news about growth rates and economic uncertainty (i.e., consumption volatility) alters perceptions regarding long-term expected growth rates and economic uncertainty
- Asset prices will be fairly sensitive to small growth rate and consumption volatility news.
- Log linearizing, they solve the model to obtain

Innovation to the pricing kernel in terms of three risks η, e, w and their market prices λ

$$m_{t+1} - E_t m_{t+1} = \lambda_{m,\eta} \overbrace{\sigma_t \eta_{t+1}}^{\text{consumption shock}} - \lambda_{m,e} \overbrace{\sigma_t e_{t+1}}^{\text{LRR shock}} - \lambda_{m,w} \sigma_w \overbrace{w_{t+1}}^{\text{shock to vol}}$$

Equity premium

$$E_t(r_{m,t+1} - r_{f,t+1}) = \beta_{m,e}\lambda_{m,e}\sigma_t^2 + \beta_{m,w}\lambda_{m,w}\sigma_w^2 - \frac{1}{2}\mathrm{var}_t(r_{m,t+1})$$

$$\mathrm{var}_t(r_{m,t+1}) = (\beta_{m,e}^2 + \varphi_d^2)\sigma_t^2 + \beta_{m,w}^2\sigma_w^2$$

Risk return relationship

$$E_t(r_{m,t+1} - r_{f,t+1}) = \tau_0 + \tau_1 \mathrm{var}_t(r_{m,t+1})$$

- Bansal et al. use data from 1928-1998. They show that
  - The model is capable of justifying the observed magnitudes of the equity premium, the risk-free rate, and the volatility of the market return and the dividend-yield.
  - It captures the volatility feedback effect, that is, the negative correlation between return news and return volatility news.
  - As in the data, dividend yields predict future returns and the volatility of returns is time-varying.
  - At plausible values for the preference parameters (IES and RRA), a reduction in economic uncertainty or better long-run growth prospects leads to a rise in the wealth–consumption and the price–dividend ratios. There is a significant negative correlation between price–dividend ratios and consumption volatility.
  - They show that about half of the variability in equity prices is due to fluctuations in expected growth rates, and the remainder is due to fluctuations in the cost of capital.
- Macro/asset pricing theory is an active area of research