

# F500: Empirical Finance

## Lecture 8: Volatility Measurement and Modelling

Oliver Linton  
obl20@cam.ac.uk

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# Outline

- 1 Why care about volatility?
- 2 Measurement/estimation of volatility
  - 1 Implied Volatility
  - 2 Realized Volatility
  - 3 Ex Ante Volatility Garch Model and variants
- 3 Some empirical studies

Reading: Linton (2019), Chapter 11

- Risk/Volatility measurement is central to finance
  - ▶ Asset pricing. Conditional CAPM

$$E_{t-1}(r_{i,t} - r_{f,t}) = \beta_{i,t} \lambda_t$$

$$\beta_{i,t} = \frac{\text{COV}_{t-1}(r_{i,t}, r_{m,t})}{\text{var}_{t-1}(r_{m,t})}$$

- ▶ Risk Management/Value at Risk

$$\text{VaR}_t(\alpha) = \mu + \underbrace{\sigma_t}_{\text{volatility forecast}} \times \underbrace{q_\alpha}_{\text{quantile of innovation}}$$

- ▶ Portfolio Allocation

$$\max_{w \in S_d} w^\top E_{t-1}(\mathbf{r}_t) \text{ s.t. } w^\top \text{var}_{t-1}(\mathbf{r}_t) w = \sigma^2$$

- ▶ Measuring market quality - highly volatile markets discourage participation

# Implied Volatility from Option Prices

- Bachelier (1900), Samuelson (1967). Suppose that stock prices  $P$  follows a geometric Brownian motion

$$d \log P(t) = \mu dt + \sigma dB(t),$$

where  $B$  is Brownian motion, i.e., for all  $t, s$

$$B(t+s) - B(t)$$

is normally distributed with mean zero and variance  $s$  with independent increments. This is a continuous time model. Prices are lognormally distributed.

- Volatility is measured by the parameter  $\sigma$  or  $\sigma^2$

- Suppose you have a European (exercisable only at maturity) call option on the stock with strike price  $X$  and time to maturity  $\tau$ . Black and Scholes (1973) showed that the option price  $C$  satisfies

$$C = P\Phi(d_1) - Xe^{-r\tau}\Phi(d_2),$$

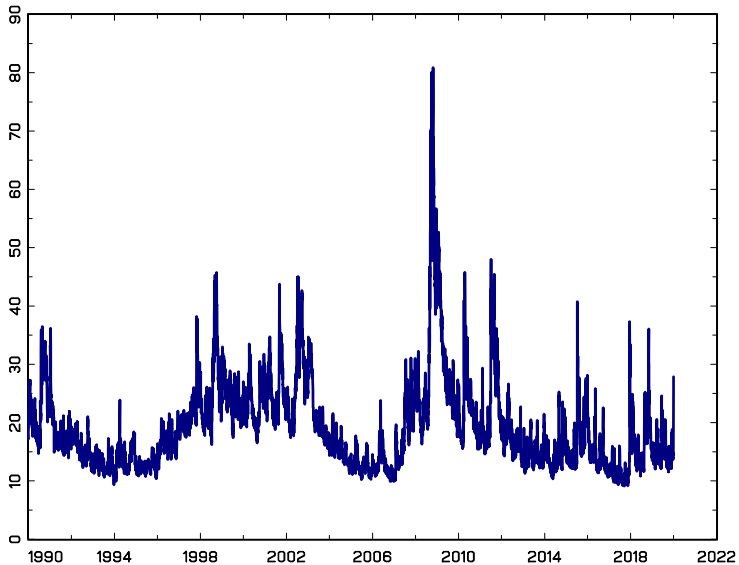
where  $r$  is the risk free rate, and:

$$d_1 = \frac{\log(P/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad ; \quad d_2 = \frac{\log(P/X) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}},$$

where  $\Phi$  is the standard normal c.d.f. Value of option increases in volatility.

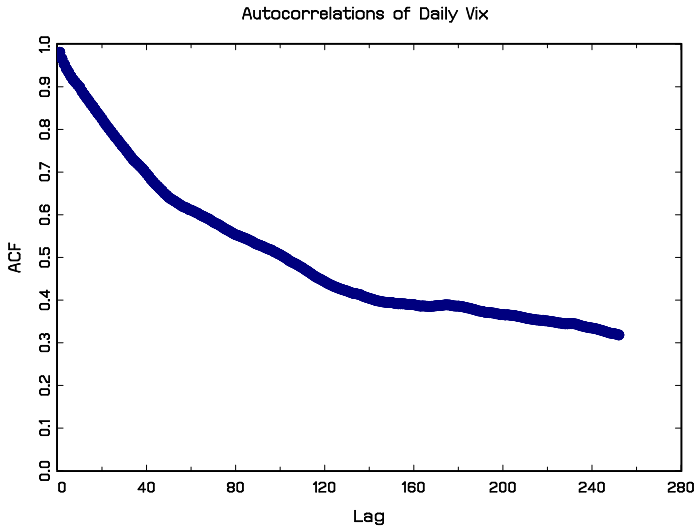
- Given observations on  $C$ ,  $P$ ,  $X$ ,  $r$ , and  $\tau$ , we can invert the relation to obtain  $\sigma^2$ . Called **implied volatility**. Can do this at every time period where we have these observations thereby generating a time series of  $\sigma^2$ . In practice, some adjustments are made.

- VIX is the ticker symbol for the Chicago Board Options Exchange Market Volatility Index, a popular measure of the implied volatility of S&P 500 index options. Often referred to as the **fear index** or the fear gauge, it represents one measure of the market's expectation of stock market volatility over the next 30 day period.
- The VIX is quoted in percentage points and translates, roughly, to the expected movement in the S&P 500 index over the next 30-day period, which is then annualized. The VIX is calculated and disseminated in real-time by the Chicago Board Options Exchange.
- It is a weighted blend of prices for a range of options on the S&P 500 index. The formula uses a kernel-smoothed estimator that takes as inputs the current market prices for all out-of-the-money calls and puts for the front month and second month expirations.  
<http://www.cboe.com/micro/vix/vixwhite.pdf>



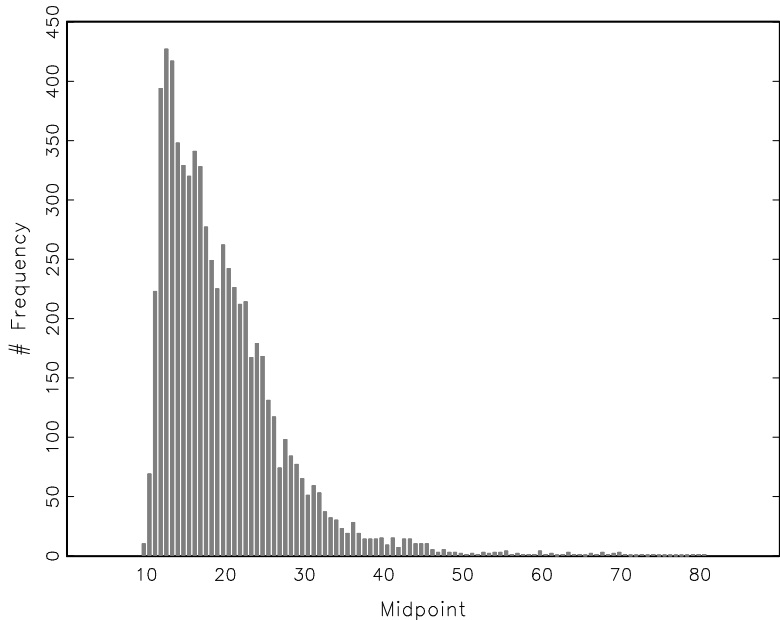
Daily Vix

Very persistent





## Skewed distribution



- This is a forward looking measure of volatility based on the positive actions of market participants.
- The interpretation of VIX as a volatility measure can be made precise in that  $VIX_t^2$  is the conditional variance of returns under the **risk neutral probability measure**.
- Martin (2017) proposes an alternative volatility measure called the SVIX, which is the conditional variance of returns under the **objective probability measure**.

## Intra period volatility

- Lets suppose that we are interested in monthly returns  $r_t$ , but we also have higher frequency data  $r_{t_j}$ ,  $j = 1, \dots, n$ , where  $n$  is the total number of observations inside each period, assumed constant for simplicity. We construct the volatility of stock over the period  $[t, t + 1]$ , denoted  $\hat{\sigma}_{t+1}^2$ , as

$$\hat{\sigma}_{t+1}^2 = \frac{1}{n} \sum_{j=1}^n r_{t_j}^2 - \left( \frac{1}{n} \sum_{j=1}^n r_{t_j} \right)^2 .$$

- This can be considered an ex-post measure of volatility, meaning it is a measure of the volatility that happened in the period  $t, t + 1$ , and not what was anticipated to happen at time  $t$ , i.e., it is not the conditional variance of returns given past information.
- In some cases people use  $\hat{\sigma}_{t+1}^2 = \sum_{j=1}^n r_{t_j}^2 / n$ , because mean daily returns are small and so their square is even smaller and can be ignored. In other cases they use an adjustment that allows for serial correlation.

- The issue with this approach is that it relies on higher frequency data and it is not clear how to interpret  $\hat{\sigma}_{t+1}^2$  in terms of plausible discrete time models of  $r$ .
- With 'continuous record asymptotics' (or "infill asymptotics") it has an interpretation, Foster and Nelson (1994). Suppose observe prices (transaction prices or even midpoint quoted prices) within a day (9am-4pm on NYSE)

| Frequency     | n (returns) |
|---------------|-------------|
| Hourly        | 7           |
| 10 mins       | 42          |
| 5 mins        | 84          |
| 1 min         | 420         |
| 10 secs       | 2520        |
| 1 sec         | 25200       |
| 1 millisecond | 25200000    |

- Nowadays  $\hat{\sigma}_{t+1}^2$  is called **realized volatility** (so long as we multiply by  $n$ ) and there is a comprehensive theory about it in infill setting, Barndorff Nielsen and Shephard (2001).

## Definition

Suppose that we observe transactions at times  $t_j, j = 0, 1, \dots, n$  and  $r_{t_j} = \log P(t_j) - \log P(t_{j-1})$ . Define the realized volatility (RV)

$$\hat{\sigma}_{t+1}^2 = \sum_{j=1}^n r_{t_j}^2.$$

Suppose that stock prices  $P$  follows a geometric Brownian motion

$$d \log P(t) = \mu dt + \sigma dB(t),$$

where  $B$  is standard Brownian motion. Suppose that  $t_j = t + j/n$

$$r_{t_j} \sim N(\mu/n, \sigma^2/n) = \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} z_j$$

where  $z_j$  are standard normal.

We have

$$\begin{aligned}\widehat{\sigma}_{t+1}^2 &= \sum_{j=1}^n r_{tj}^2 = \sum_{j=1}^n \left( \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} z_j \right)^2 = \sigma^2 \frac{1}{n} \sum_{j=1}^n z_j^2 + \frac{\mu^2}{n} + \frac{1}{n} \frac{2\mu}{\sqrt{n}} \sum_{j=1}^n z_j \\ &= \sigma^2 + \sigma^2 \left( \frac{1}{n} \sum_{j=1}^n z_j^2 - 1 \right) + \frac{\mu^2}{n} + \frac{1}{n} \frac{2\mu}{\sqrt{n}} \sum_{j=1}^n z_j\end{aligned}$$

## Theorem

Therefore, it follows that as  $n \rightarrow \infty$ ,

$$\widehat{\sigma}_{t+1}^2 \xrightarrow{P} \sigma^2$$

$$\sqrt{n} \left( \widehat{\sigma}_{t+1}^2 - \sigma^2 \right) \implies N(0, 2\sigma^4)$$

- A central limit theorem continues to hold under much more general conditions, specifically for **Diffusion process**

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

where ( $X = \log P$ ) the parameter of interest is the "quadratic variation" of  $X$ , specifically

$$QV_{t,t+1} = \int_t^{t+1} \sigma^2(X_s)ds,$$

which is a stochastic quantity. Realized volatility consistently estimates this quantity (drift not important). Asymptotic mixed normality.

- Also true for **Stochastic volatility models** such as

$$dX_t = \mu_t dt + \sigma_t dW_t$$

$$d\sigma_t = m_t dt + v_t dW_t$$

such as Heston model.

## Volatility Measures using only open, close, high and low

- Yahoo, Bloomberg etc all report the daily opening price, closing price, the intraday high price, and the intraday low price:  $P_O, P_C, P_H, P_L$ .
- Most authors work with the daily closing price and returns as we have described them have been computed this way.

### Definition

A simple measure of volatility is

$$V_t^{HL} = \frac{P_{Ht} - P_{Lt}}{P_{Lt}}.$$

- Actually, can replace the denominator by  $P_{Ct}$  for example without much change in the result. This also has an interpretation inside continuous time models.



## Definition

The Parkinson (1980) estimator

$$V_t^P = \frac{(\log P_{Ht} - \log P_{Lt})^2}{4 \log 2}$$

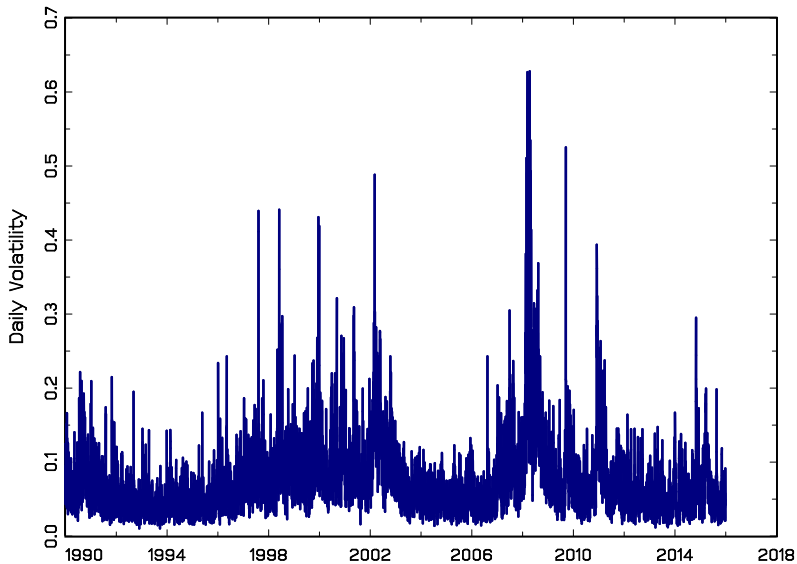
- This consistently estimates the parameter  $\sigma^2$  of the Brownian motion process. But it is an inefficient estimator of  $\sigma^2$  under the Brownian motion model assumption (if you had all the returns).
- The Garman and Klass (1980), Rogers and Satchell (1991) estimators provide some improvement in efficiency and correct for a drift:

$$V_t^{GK} = 0.5 \left( \ln P_t^H - \ln P_t^L \right)^2 - (2 \ln 2 - 1) \left( \ln P_t^C - \ln P_t^O \right)^2$$

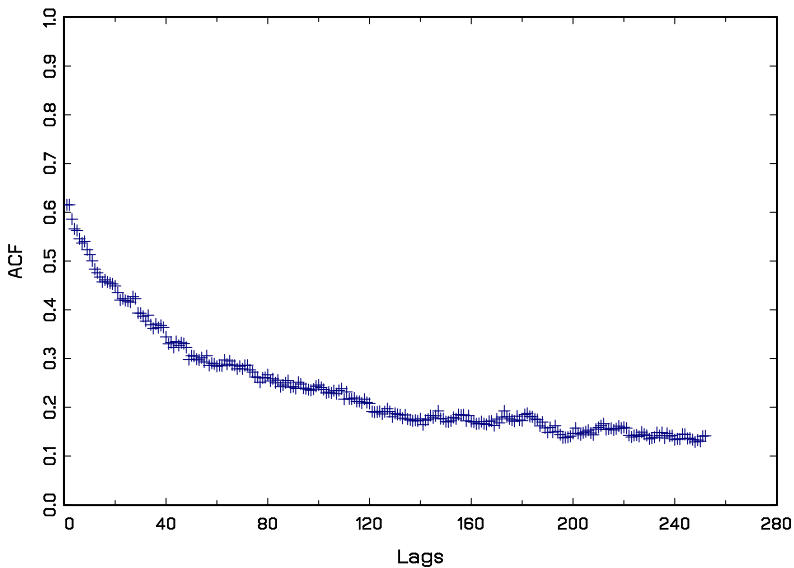
$$V_t^{RS} = (\ln P_t^H - \ln P_t^C)(\ln P_t^H - \ln P_t^O) + (\ln P_t^L - \ln P_t^C)(\ln P_t^L - \ln P_t^O)$$

Chou et al. (2009) for a discussion of range based volatility estimators.

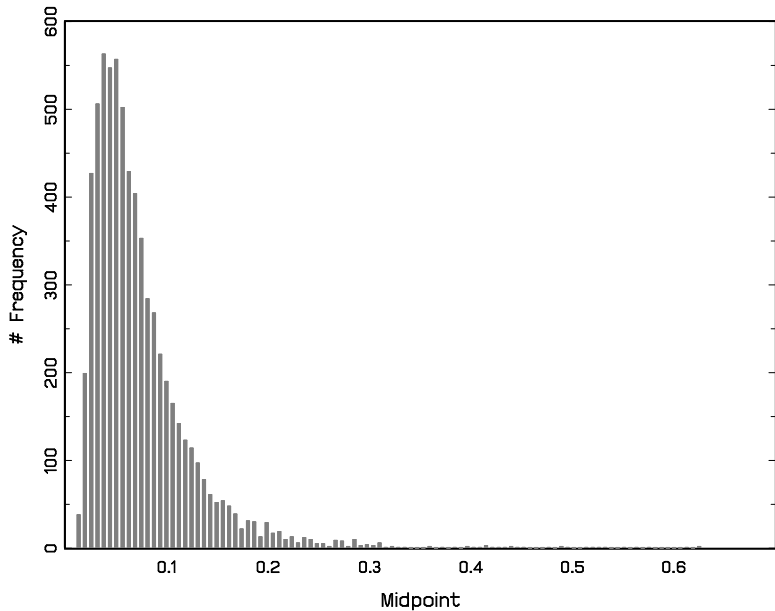
# SP500



## Autocorrelations



## Skewed, long right tail



# Time Series GARCH Models

- Empirically, squared high-frequency returns have strong positive autocorrelation
- Interested in ex ante measures of volatility
- Investors wish to trade-off risk versus return based on current knowledge.

## Definition

Engle (1982), Bollerslev (1986) Generalized AutoRegressive Conditional Heteroskedasticity GARCH(1,1) model

$$\begin{aligned}r_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \omega + \beta \sigma_{t-1}^2 + \gamma r_{t-1}^2,\end{aligned}$$

where  $\varepsilon_t$  is i.i.d normal with mean zero and variance one.

- Provided  $\omega > 0$  and  $\beta, \gamma \geq 0$ , then  $\sigma_t^2 > 0$  with probability one and

$$\sigma_t^2 = \text{var}(r_t | \mathcal{F}_{t-1}),$$

where  $\mathcal{F}_{t-1}$  is past information.

- The process  $r_t$  is **weakly stationary** provided

$$\beta + \gamma < 1$$

and has finite unconditional variance

$$\sigma^2 = E(\sigma_t^2) = \frac{\omega}{1 - \beta - \gamma}.$$

- Can write

$$\begin{aligned}\sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \gamma r_{t-1}^2 \\ &= \omega + \beta\omega + \gamma r_{t-1}^2 + \beta\gamma r_{t-2}^2 + \beta^2\sigma_{t-2}^2 \\ &= \frac{\omega}{1-\beta} + \gamma \sum_{j=1}^{\infty} \beta^{j-1} r_{t-j}^2.\end{aligned}$$

- $\sigma_t^2$  depends on all past squared returns. The weighting  $\beta^{j-1}$  of each lagged squared return declines geometrically fast

## Dependence of the process

- We can write the process as an ARMA(1,1) in  $r_t^2$ . Specifically, note that

$$\begin{aligned}r_t^2 &= \sigma_t^2 \varepsilon_t^2 \\ &= \sigma_t^2 + \sigma_t^2 (\varepsilon_t^2 - 1) \\ &= \omega + \beta \sigma_{t-1}^2 + \gamma r_{t-1}^2 + \sigma_t^2 (\varepsilon_t^2 - 1) \\ &= \omega + (\beta + \gamma) r_{t-1}^2 + \eta_t + \beta \eta_{t-1},\end{aligned}$$

where  $\eta_t = r_t^2 - \sigma_t^2 = \sigma_t^2 (\varepsilon_t^2 - 1)$  is a mean zero innovation uncorrelated with its past, albeit heteroskedastic, i.e., an *MDS*.

- This guarantees some dependence in  $r_t^2$ . In particular, we have positive dependence, i.e.,

$$\text{cov}(r_t^2, r_{t-j}^2) > 0$$

for all  $j$ . This can be observed in the data. Also  $\text{cov}(\sigma_t^2, \sigma_{t-j}^2) > 0$

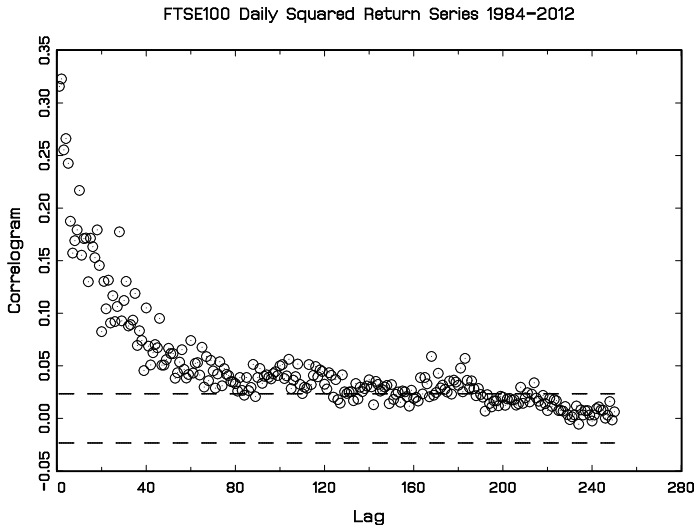


- Persistence of shocks to volatility is measured by  $\beta + \gamma$
- In practice, estimated parameters lie close to the boundary of this region i.e.,  $\beta + \gamma \sim 1$ .
- The IGARCH model has

$$\beta + \gamma = 1.$$

In this case, the process  $r_t$  is strongly stationary but is not covariance stationary [the unconditional variance is infinite].

We can see that the correlogram of  $r_t^2$  supports the GARCH model for many financial time series



## Theorem

The marginal distribution of  $r_t$  will be heavy tailed even if  $\varepsilon_t$  is standard normal. Suppose that  $\varepsilon_t$  is standard normal, then

$$\frac{E(\varepsilon_t^4)}{(E(\varepsilon_t^2))^2} = 3.$$

Furthermore, let's suppose that the kurtosis and all fourth order moments are well defined and time invariant (the process  $r_t$  is stationary). Then we have (provided  $\beta + \gamma < 1$ )

$$\frac{E r_t^4}{(E r_t^2)^2} = \frac{3(1 - (\beta + \gamma)^2)}{1 - 2\gamma^2 - (\beta + \gamma)^2} > 3$$

Example S&P500 daily stock index return series from 1955-2002; Eviews.

# Computing GARCH Estimates

- The log-likelihood for a general-variant Garch model given normal  $\varepsilon_t$  has a simple form

$$\ell_T(\omega, \beta, \gamma) = c - \frac{1}{2} \sum_{t=1}^T \frac{r_t^2}{\sigma_t^2(\omega, \beta, \gamma)} - \frac{1}{2} \sum_{t=1}^T \log \sigma_t^2(\omega, \beta, \gamma)$$

- Use numerical techniques to minimize minus the summed log likelihood of the sample
- The oft-observed near-flatness of the Garch likelihood surface means that  $T$  must be large for reliable estimates

## Parametric Estimation from EViews

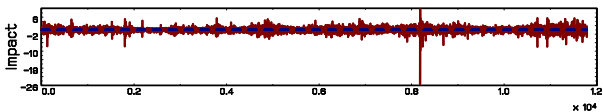
|          | Daily                        | Weekly                       | Monthly                    |
|----------|------------------------------|------------------------------|----------------------------|
| $\rho_1$ | 0.134457<br>(0.009531)       | 0.002871<br>(0.021709)       | 0.020311<br>(0.049513)     |
| $\rho_2$ | -0.027715<br>(0.009327)      | 0.032247<br>(0.022462)       | -0.058747<br>(0.045839)    |
| $\omega$ | $6.46E-07$<br>( $7.28E-08$ ) | $1.14E-05$<br>( $2.50E-06$ ) | 0.000103<br>( $4.63E-05$ ) |
| $\beta$  | 0.913948<br>(0.002366)       | 0.845708<br>(0.014920)       | 0.870139<br>(0.040540)     |
| $\gamma$ | 0.082395<br>(0.001757)       | 0.131007<br>(0.012950)       | 0.074298<br>(0.027528)     |

Note: Standard errors in parentheses. These estimates are for the raw data series and refer to the AR(2)-GARCH(1,1) model

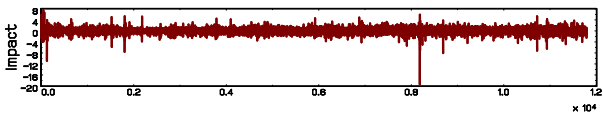
$$r_t = c + \rho_1 r_{t-1} + \rho_2 r_{t-2} + \underbrace{\varepsilon_t \sigma_t}_{u_t}$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma u_{t-1}^2$$

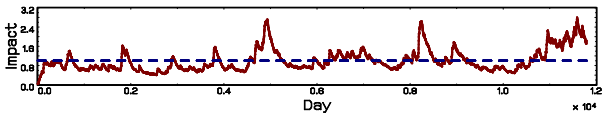
Daily S&P500 Data, 1955-2002



Standardized Residuals



Conditional Variance



# Leverage Effect and Volatility Feedback

## Definition

Leverage hypothesis is that negative returns lower equity price thereby increasing corporate leverage, thereby increasing equity return volatility. Causality runs from prices to volatility. Black (1976) and Christie (1982)

## Definition

Volatility feedback effect. If volatility is priced, an anticipated increase in volatility would raise the required rate of return, in turn requiring an immediate stock-price decline to allow for higher future returns. Causality runs from volatility to prices. French, Schwert, and Stambaugh (1987) and Campbell and Hentschel (1992).

Garch linear-quadratic formulation is theoretically arbitrary, and it does not capture either phenomenon.

- We have  $r = |r| \times \text{sign}(r)$ . Therefore, we could measure asymmetry by  $\text{cov}(\sigma_t^2, \text{sign}(r_{t-j}))$ ; equivalently  $\text{cov}(\sigma_t^2, r_{t-j})$
- In a pure GARCH model with mean zero returns with normal innovations

$$\begin{aligned} \text{cov}(\sigma_t^2, r_{t-j}) &= \gamma \sum_{k=1}^{\infty} \beta^{k-1} \text{cov}(r_{t-k}^2, r_{t-j}) \\ &= \gamma \sum_{k=1}^{\infty} \beta^{k-1} E(\sigma_{t-k}^2 \sigma_{t-j}) E(\varepsilon_{t-k}^2 \varepsilon_{t-j}) = 0. \end{aligned}$$

This is zero because

$$E(\varepsilon_{t-k}^2 \varepsilon_{t-j}) = 0$$

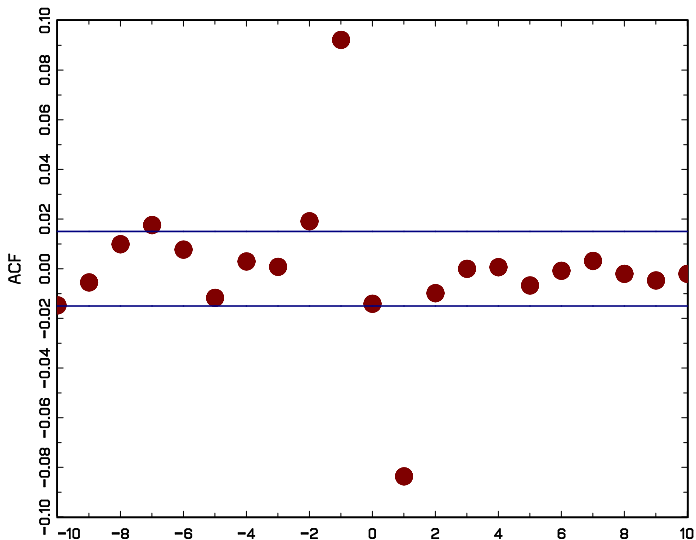
for all  $k, j$ . If  $k \neq j$  this is true by independence of  $\varepsilon_s$  over  $s$ , when  $j = k$  this is true if  $\varepsilon_t$  is symmetrically distributed about zero, e.g., normal distribution).

- Therefore, the classic GARCH models rule out leverage effect.

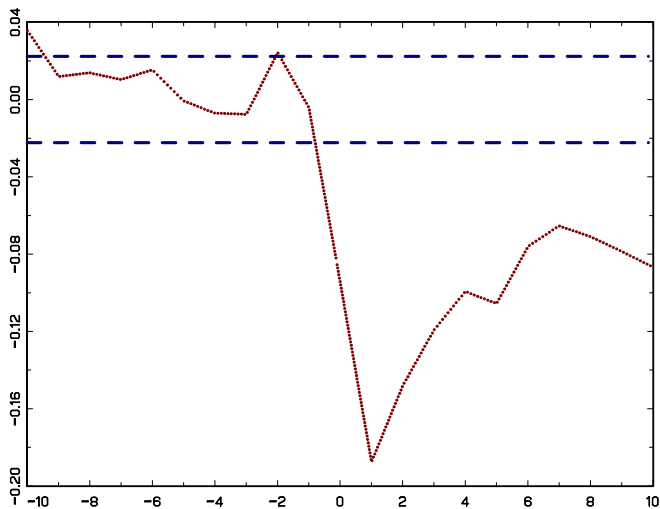


- Regarding the leverage effect, the magnitude of the effect of a decline in current prices on future volatilities appears too large to be explained solely by changes in financial leverage [see, e.g., Figlewski and Wang (2001)]. Furthermore, counter to a leverage-based explanation, the asymmetry is generally larger for aggregate market index returns than that for individual stocks [see, e.g., Kim and Kon (1994), Tauchen, Zhang, and Liu (1996), and Andersen et al. (2001)].
- Bekaert and Wu (2000) and Wu (2001) argued that the volatility feedback effect dominates the leverage effect empirically.
- However, many other studies [see, e.g., Nelson (1991), Engle and Ng (1993), and Glosten, Jagannathan, and Runkle (1993)] have found that volatility increases more following negative returns than positive returns and that the relationship between expected returns and volatility is insignificant, or even negative empirically.

Evidence for asymmetric effect S&P500 Daily return cross autocovariance  $\text{cov}(r_t^2, r_{t-j})$ ,  $j = -10, \dots, 10$ . Both directions significant at one lag



Shows acf of intraday return (open to close) with the Parkinson volatility estimator. Granger causality seems to go from returns to volatility not vice versa, and effect is negative.



# Asymmetric GARCH

- We write

$$x^2 = \overbrace{x^2 \mathbf{1}(x > 0)}^{\text{good news}} + \overbrace{x^2 \mathbf{1}(x \leq 0)}^{\text{bad news}}$$

- Glosten, Jeganathan and Runkle (1994)

$$\begin{aligned}\sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \gamma r_{t-1}^2 + \delta r_{t-1}^2 \mathbf{1}(r_{t-1} < 0) \\ &= \omega + \beta\sigma_{t-1}^2 + \psi_+ r_{t-1}^2 \mathbf{1}(r_{t-1} \geq 0) + \psi_- r_{t-1}^2 \mathbf{1}(r_{t-1} < 0),\end{aligned}$$

where  $\gamma = \psi_+ + \psi_-$  and  $\delta = \psi_-$ . Equivalent parameterizations.

## Parametric Estimation from EViews

|                        | Daily                    | Weekly                 | Monthly                 |
|------------------------|--------------------------|------------------------|-------------------------|
| $\rho_1$               | 0.138788<br>(0.009524)   | 0.007065<br>(0.022000) | 0.014661<br>(0.045131)  |
| $\rho_2$               | -0.01906<br>(0.009449)   | 0.051815<br>(0.022044) | -0.018694<br>(0.045083) |
| $\omega (\times 1000)$ | 0.0000721<br>(0.0000064) | 0.00130<br>(0.000242)  | 0.862000<br>(0.249000)  |
| $\beta$                | 0.920489<br>(0.002243)   | 0.850348<br>(0.015580) | 0.442481<br>(0.176365)  |
| $\gamma$               | 0.034018<br>(0.002613)   | 0.047885<br>(0.013504) | -0.076662<br>(0.042047) |
| $\delta$               | 0.078782<br>(0.003302)   | 0.140013<br>(0.020349) | 0.266916<br>(0.094669)  |

Note: Standard errors in parentheses. These estimates are for the raw S&P500 data series and refer to the AR(2)-AGARCH(1,1) model

$$r_t = c + \rho_1 r_{t-1} + \rho_2 r_{t-2} + \overbrace{\varepsilon_t \sigma_t^{u_t}}^{u_t}$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma u_{t-1}^2 + \delta u_{t-1}^2 \mathbf{1}(u_{t-1} < 0)$$

# GARCH in Mean

- In intertemporal general equilibrium, the risk premium associated with equities might increase when volatility increases
- The Garch-M model assumes that the risk premium is linear in known function of volatility

$$\mu_t = E(r_t | \mathcal{F}_{t-1}) = \alpha_0 + \alpha_1 g(\sigma_t^2), \quad g(x) = x, \sqrt{x}, \ln x$$

$$\sigma_t^2 = \text{var}(r_t | \mathcal{F}_{t-1}) = \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma (r_{t-1} - \mu_{t-1})^2$$

## Parametric Estimation from EViews

|          | Daily                        | Weekly                       | Monthly                |
|----------|------------------------------|------------------------------|------------------------|
| $\alpha$ | 0.081504<br>(0.029699)       | 0.121757<br>(0.076905)       | 0.415873<br>(0.327167) |
| $\omega$ | $6.49E-07$<br>( $7.48E-08$ ) | $1.13E-05$<br>( $2.53E-06$ ) | 0.000125<br>(0.072803) |
| $\beta$  | 0.916160<br>(0.002356)       | 0.846601<br>(0.014707)       | 0.858988<br>(0.044015) |
| $\gamma$ | 0.079801<br>(0.001737)       | 0.130387<br>(0.012697)       | 0.072803<br>(0.027614) |

Note: Standard errors in parentheses. These estimates are for the raw S&P500 data series and refer to the GARCH(1,1) in mean model

$$r_t = c + \alpha\sigma_t + \varepsilon_t\sigma_t$$

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \gamma u_{t-1}^2$$

# Nonstationarity

- Recently, a criticism of GARCH processes has come to the fore, namely their usual assumption of stationarity.
- By taking  $\beta + \gamma \geq 1$  one can have nonstationary processes, but at the cost of non-existence of unconditional variance.
- Instead, maybe the coefficients change over time, thus

$$\sigma_t^2 = \omega_t + \beta_t \sigma_{t-1}^2 + \gamma_t r_{t-1}^2$$

with  $\beta_t + \gamma_t < 1$



|                 | 50-60   | 60-70   | 70-80   | 80-90   | 90-00  | 00-09   |
|-----------------|---------|---------|---------|---------|--------|---------|
| $c$             | 0.0213  | 0.0019  | -0.0106 | 0.0111  | 0.0043 | -0.0234 |
| $\rho_1$        | 0.1584  | 0.2229  | 0.2429  | 0.0635  | 0.0603 | -0.0814 |
| $\rho_2$        | -0.0977 | -0.0288 | -0.0563 | -0.0033 | 0.0120 | -0.0445 |
| $\omega$        | 0.0425  | 0.0166  | 0.0039  | 0.0605  | 0.0132 | 0.0121  |
| $\beta$         | 0.8330  | 0.8086  | 0.9543  | 0.8620  | 0.9230 | 0.9416  |
| $\gamma$        | 0.0584  | 0.0574  | 0.0073  | 0.0362  | 0.0016 | -0.0179 |
| $\delta$        | 0.0692  | 0.2031  | 0.0691  | 0.0980  | 0.1264 | 0.1278  |
| $R^2$           | 0.0165  | 0.0320  | 0.0515  | 0.0025  | 0.0000 | 0.0171  |
| $mper$          | 0.0607  | 0.1941  | 0.1866  | 0.0602  | 0.0723 | -0.1259 |
| $vper$          | 0.9260  | 0.9676  | 0.9962  | 0.9472  | 0.9878 | 0.9876  |
| $\mu_{year}$    | 0.1199  | 0.0714  | 0.0346  | 0.0939  | 0.0768 | 0.0161  |
| $\sigma_{year}$ | 0.1144  | 0.1080  | 0.1520  | 0.1616  | 0.1571 | 0.1492  |

$$r_t = c + \rho_1 r_{t-1} + \rho_2 r_{t-2} + \varepsilon_t \sigma_t$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma u_{t-1}^2 + \delta u_{t-1}^2 \mathbf{1}(u_{t-1} < 0)$$

## Multivariate volatility models

- Multivariate models treat the whole covariance matrix as time-varying. Define

$$\Sigma_t = E(r_t r_t^\top | \mathcal{F}_{t-1}) = (\text{cov}_{t-1}(r_{it}, r_{jt}))_{i,j},$$

for some  $n \times 1$  vector of mean zero series  $r_t$ . Bollerslev et al. (1988)

$$h_t = \text{vech}(\Sigma_t) = A + B h_{t-1} + C \text{vech}(r_{t-1} r_{t-1}^\top),$$

where  $A$  is an  $n(n+1)/2 \times 1$  vector, while  $B, C$  are  $n(n+1)/2 \times n(n+1)/2$  matrices.

- The cross-section is naturally large with asset returns data. Naive Garch extension has  $\frac{n^2(n+1)^2}{2} + \frac{n(n+1)}{2}$  so with modest  $n = 1000$  this requires estimating five hundred billion ( $5 * 10^{11}$ ) parameters!
- Relatively flat Garch log likelihood function requires  $\frac{T}{\#parameters}$  to be large for reliable estimation

## Parsimony in Multivariate GARCH Models

- Factor GARCH model where the factors  $f_t$  are observed (centred and orthogonal) portfolios or macrovariables

$$r_{it} = b_i^\top f_t + \varepsilon_{it}$$

$$f_{kt} = \sigma_{k,t} e_{kt}, \quad k = 1, \dots, K$$

$$\sigma_{k,t}^2 = \omega_k + \beta_k \sigma_{k,t-1}^2 + \gamma_k f_{k,t-1}^2,$$

where  $e_{kt}$  are iid with mean zero and variance one and mutually independent.

- Then

$$\text{cov}(r_{it}, r_{jt} | \mathcal{F}_{t-1}) = \sigma_{ij,t} = b_i^\top \text{diag}\{\sigma_{1,t}^2, \dots, \sigma_{K,t}^2\} b_j + s_{ij}$$

Strict factor model has  $s_{ij} = \text{cov}(\varepsilon_{it}, \varepsilon_{jt}) = 0$  if  $i \neq j$

- If the factors are observed, then estimate  $b_i$  by time series regression. Estimate GARCH(1,1) parameters using the factor time series one by one

# CCC Model

## Definition

For  $i, j = 1, \dots, n$

$$r_{it} = \sigma_{i,t} \varepsilon_{it}$$

$$\sigma_{i,t}^2 = \omega_{ii} + \beta_i \sigma_{i,t-1}^2 + \gamma_i r_{i,t-1}^2$$

$\varepsilon_{it}$  iid with  $E\varepsilon_{it} = 0$ ,  $E\varepsilon_{it}^2 = 1$ ,  $E\varepsilon_{it}\varepsilon_{jt} = \rho_{ij}$ ,

$$\sigma_{ij,t} = \rho_{ij} (\sigma_{i,t}^2 \sigma_{j,t}^2)^{1/2}$$

- Estimate univariate Garch models using maximum likelihood, then calculate the sample correlation matrix of the standardised outcomes
- DCC model of Engle and Sheppard (2002), see corrected version [cDCC papers.ssrn.com/sol3/papers.cfm?abstract\\_id=1507743](https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1507743)

## Schwert (1989,2010)

- He examines monthly US volatility computed as sum of squared daily returns for 1885-1987 and a regression model approach like GARCH for 1857-1985.
- Main findings
  - ▶ The average level of volatility is higher during (NBER dated) recessions
  - ▶ The level of volatility during the great Depression was very high
  - ▶ The effect of financial leverage on volatility is small
  - ▶ There is weak evidence that macroeconomic volatility can help to predict financial asset volatility and stronger evidence for the reverse prediction
  - ▶ The number of trading days in the month is positively related to stock volatility (Trading days per year NYSE 252, LSE 255 (but 24 Dec is half day))
  - ▶ Share trading volume growth is positively related to stock volatility

# French and Roll (1986) Volatility over weekend and holidays

- Calendar time hypothesis: Variance is proportional to calendar time
- Trading time hypothesis: Variance is proportional only to the trading time

Typical trading day may be 8 hours long out of 24 hours (say 8-4).  
Weekend, Friday close to Monday open contains 64 hours.  
Suppose that hourly stock returns satisfy

$$r_t \sim \mu_h, \sigma_h^2$$

Then daily returns (open to close) satisfy

$$r_t \sim 8\mu_h, 8\sigma_h^2$$

Monday open from Friday close returns satisfy

$$r_t \sim \begin{cases} 64\mu_h, 64\sigma_h^2 & \text{Calendar time} \\ 0 & \text{Trading Time} \end{cases}$$

- They calculate variance as the average of squared returns over stocks and over the relevant period

$$\mathit{varperhour}_{\mathit{monday}} = \frac{1}{n_{\mathit{monday}}} \sum_{\mathit{mondays}} \frac{(p_{\mathit{mclose}} - p_{\mathit{mopen}})^2}{8}$$

$$\mathit{varperhour}_{\mathit{weekend}} = \frac{1}{n_{\mathit{weekend}}} \sum_{\mathit{weekends}} \frac{(p_{\mathit{mopen}} - p_{\mathit{fclose}})^2}{64}$$

- They find that per hour return variance is 70 times larger during a trading hour than during a weekend hour

Is this because:

- 1 Volatility is caused by public information which is more likely to arrive during normal business hours
- 2 Volatility is caused by private information which affects prices when informed investors trade
- 3 Volatility is caused by pricing errors that occur during trading

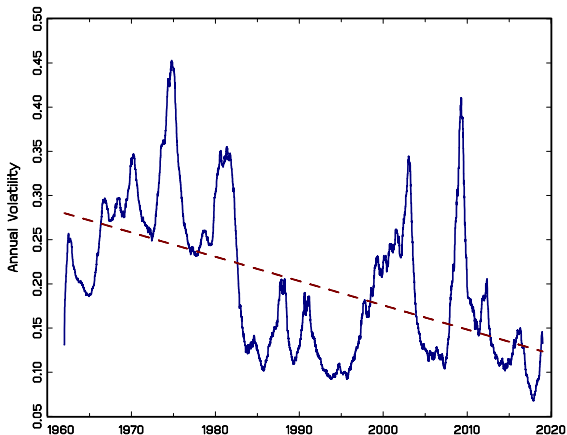
They find:

- There are some pricing errors (evidenced from autocorrelation) due to microstructure and misspricing issues but most is caused by information release
- To distinguish between public and private information (explanations 1 and 2) they use the fact that in 1968, NYSE was closed every wednesday because of "paperwork crisis", but otherwise was a regular business day. [Explanation 2 is their main story.](#)



## Is volatility trending upwards?

Shows rolling window annual median of daily RS volatility estimator for S&P500 with trend line



FTSE100 Top 20 Most Volatile days since 1984 (- means  $P_C < P_O$ , + means  $P_C > P_O$ )

| Date     | Volatility |          |        |
|----------|------------|----------|--------|
| 19871020 | 0.131-     |          |        |
| 19871022 | 0.115-     |          |        |
| 20081010 | 0.112-     | 20081013 | 0.076+ |
| 19971028 | 0.096-     | 20010921 | 0.076- |
| 20081024 | 0.096-     | 20081029 | 0.075+ |
| 20081006 | 0.094-     | 20110809 | 0.074+ |
| 20081008 | 0.094-     | 20090114 | 0.074- |
| 20080919 | 0.093+     | 20080122 | 0.074+ |
| 20081124 | 0.090+     | 20020715 | 0.071- |
| 20081015 | 0.084-     | 20081016 | 0.070- |
| 19871019 | 0.081-     |          |        |
| 20020920 | 0.080+     |          |        |

## S&P500 Top 20 Most Volatile days since 1960

| Date     | Volatility |          |        |
|----------|------------|----------|--------|
| 19871019 | 0.257-     |          |        |
| 19871020 | 0.123+     | 20081201 | 0.089+ |
| 20081010 | 0.107-     | 19620529 | 0.089+ |
| 20081009 | 0.106-     | 19871021 | 0.087+ |
| 20081113 | 0.104+     | 20081016 | 0.087+ |
| 20081028 | 0.101+     | 20081006 | 0.085- |
| 20081015 | 0.100-     | 20081022 | 0.085- |
| 20081120 | 0.097-     | 20020724 | 0.081+ |
| 20081013 | 0.094+     | 19980831 | 0.080- |
| 20080929 | 0.093-     |          |        |
| 19871026 | 0.092-     |          |        |
| 20100506 | 0.090-     |          |        |

- Both markets dominated by 2008 and 1987
- US market a little more volatile than UK
  - ▶ perhaps explained by more innovation? perhaps not?
- Circuit breakers now limit the worst case, or perhaps spread it out over several days