F500: Empirical Finance
Lecture 8: Volatility Measurement and Modelling

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Outline

1. Why care about volatility?
2. Measurement/estimation of volatility
   1. Implied Volatility
   2. Realized Volatility
   3. Ex Ante Volatility Garch Model and variants
3. Some empirical studies

Reading: Linton (2019), Chapter 11

- Risk/Volatility measurement is central to finance
  - Asset pricing. Conditional CAPM

$$E_{t-1} r_{i,t} - r_f = \beta_{i,t} \lambda_t$$

$$\beta_{i,t} = \frac{\mathrm{cov}_{t-1}(r_{i,t}, r_{m,t})}{\mathrm{var}_{t-1}(r_{m,t})}$$

  - Risk Management/Value at Risk

$$VaR_t(\alpha) = \mu + \overbrace{\sigma_t}^{\text{volatility forecast}} \times \overbrace{q_\alpha}^{\text{quantile of innovation}}$$

  - Portfolio Allocation

$$\max_{w \in S_d} w^\intercal E_{t-1}(\mathbf{r}_t) \text{ s.t. } w^\intercal \mathrm{var}_{t-1}(\mathbf{r}_t) w = \sigma^2$$

  - Measuring market quality - highly volatile markets discourage participation

Implied Volatility from Option Prices

- Bachelier (1900), Samuelson (1967). Suppose that stock prices P follows a geometric Brownian motion

$$d \log P(t) = \mu dt + \sigma dB(t),$$

where B is Brownian motion, i.e., for all t, s

$$B(t+s) - B(t)$$

is normally distributed with mean zero and variance s with independent increments. This is a continuous time model. Prices are lognormally distributed.

- Volatility is measured by the parameter σ or σ^2

- Suppose you have a European (exercisable only at maturity) call option on the stock with strike price X and time to maturity τ. Black and Scholes (1973) showed that the option price C satisfies

$$C = P\Phi(d_1) - Xe^{-r\tau}\Phi(d_2),$$

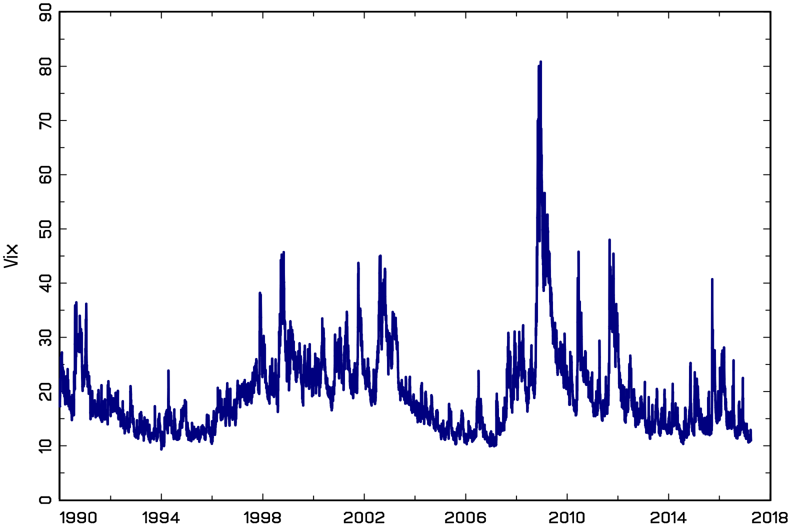
where r is the risk free rate, and:

$$d_1 = \frac{\log(P/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad ; \quad d_2 = \frac{\log(P/X) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}},$$

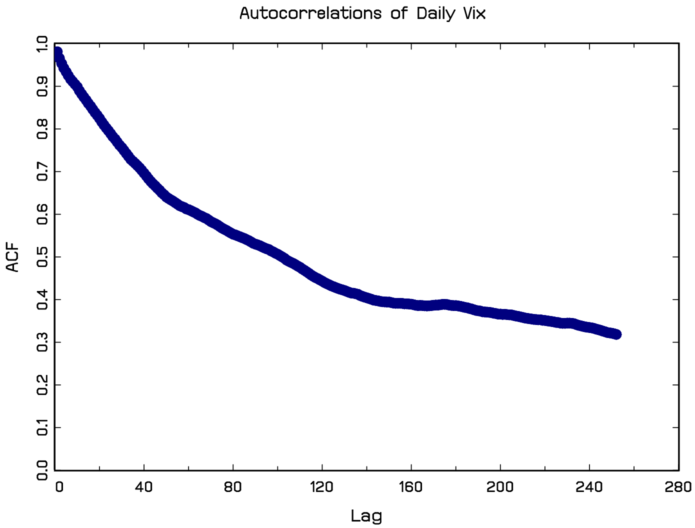
where Φ is the standard normal c.d.f. Value of option increases in volatility.

- Given observations on C, P, X, r, and τ, we can invert the relation to obtain σ^2. Called **implied volatility**. Can do this at every time period where we have these observations thereby generating a time series of σ^2. In practice, some adjustments are made.

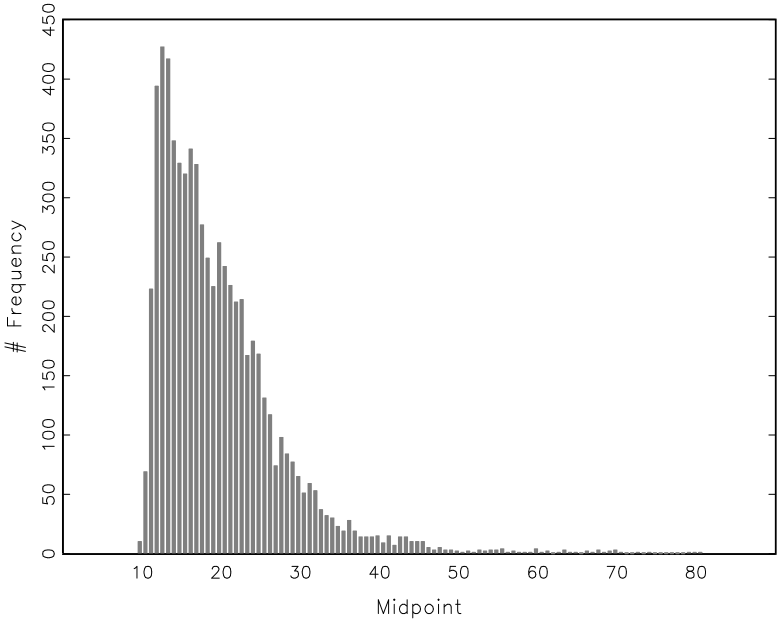
- VIX is the ticker symbol for the Chicago Board Options Exchange Market Volatility Index, a popular measure of the implied volatility of S&P 500 index options. Often referred to as the **fear index** or the fear gauge, it represents one measure of the market's expectation of stock market volatility over the next 30 day period.
- The VIX is quoted in percentage points and translates, roughly, to the expected movement in the S&P 500 index over the next 30-day period, which is then annualized. The VIX is calculated and disseminated in real-time by the Chicago Board Options Exchange.
- It is a weighted blend of prices for a range of options on the S&P 500 index. The formula uses a kernel-smoothed estimator that takes as inputs the current market prices for all out-of-the-money calls and puts for the front month and second month expirations. http://www.cboe.com/micro/vix/vixwhite.pdf


Vix
0
10
20
30
40
50
60
70
80
90
1990
1994
1998
2002
2006
2010
2014
2018

Very persistent



Skewed distribution (measures on a log-log scale)



- This is a forward looking measure of volatility.
- The interpretation of VIX as a volatility measure can be made precise in that VIX_t^2 is the conditional variance of returns under the **risk neutral probability measure**. Martin (2017) proposes an alternative volatility measure called the SVIX, which is the conditional variance of returns under the **objective probability measure**.

Intra period volatility

- Lets suppose that we are interested in monthly returns r_t, but we also have higher frequency data r_{t_j}, $j = 1, \ldots, n$, where n is the total number of observations inside each period, assumed constant for simplicity. We construct the volatility of stock at time $t+1$, $\widehat{\sigma}^2_{t+1}$, as

$$\widehat{\sigma}^2_{t+1} = \frac{1}{n}\sum_{j=1}^{n} r_{t_j}^2 - \left(\frac{1}{n}\sum_{j=1}^{n} r_{t_j}\right)^2 .$$

- This can be considered an ex-post measure of volatility, meaning it is a measure of the volatility that happened in the period $t, t+1$, and not what was anticipated to happen at time t, i.e., it is not the conditional variance of returns given past information.
- In some cases people use $\widehat{\sigma}^2_{t+1} = \sum_{j=1}^{n} r_{t_j}^2 / n$, because mean daily returns are small and so their square is even smaller and can be ignored. In other cases they use an adjustment that allows for serial correlation.

- The issue with this approach is that it relies on higher frequency data and it is not clear how to interpret $\widehat{\sigma}^2_{t+1}$ in terms of plausible discrete time models of r.
- With ‘continuous record asymptotics’ (or "infill asymptotics") it has an interpretation, Foster and Nelson (1994). Suppose observe prices (transaction prices or even midpoint quoted prices) within a day (9am-4pm on NYSE)

Frequency	n (returns)
Hourly	7
10 mins	42
5 mins	84
1 min	420
10 secs	2520
1 sec	25200
1 millisecond	25200000

Nowadays this is called **realized volatility** (so long as we multiply by n) and there is a comprehensive theory about it, Barndorff Nielsen and Shephard (2001).

Definition

Suppose that we observe transactions at times t_j, $j = 0, 1, \ldots, n$ and $r_{t_j} = \log P(t_j) - \log P(t_{j-1})$. Define the realized volatility (RV)

$$\widehat{\sigma}^2_{t+1} = \sum_{j=1}^{n} r^2_{t_j}.$$

Suppose that stock prices P follows a geometric Brownian motion

$$d \log P(t) = \mu dt + \sigma dB(t),$$

where B is standard Brownian motion. Suppose that $t_j = j/n$

$$r_{t_j} \sim N(\mu/n, \sigma^2/n) = \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} z_j$$

where z_j are standard normal.

We have

$$\widehat{\sigma}^2_{t+1} = \sum_{j=1}^{n} r^2_{t_j} = \sum_{j=1}^{n} \left(\frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} z_j\right)^2 = \sigma^2 \frac{1}{n} \sum_{j=1}^{n} z_j^2 + \frac{\mu^2}{n} + \frac{1}{n} \frac{2\mu}{\sqrt{n}} \sum_{j=1}^{n} z_j,$$

Theorem

Therefore, it follows that as $n \to \infty$,

$$\widehat{\sigma}^2_{t+1} \to \sigma^2 \ \textit{wp1}$$

$$\sqrt{n}\left(\widehat{\sigma}^2_{t+1} - \sigma^2\right) \Longrightarrow N(0, 2\sigma^4)$$

- This result continues to hold under much more general conditions, specifically for **Diffusion process**

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

where ($X = \log P$) the parameter of interest is the "quadratic variation" of X, specifically

$$QV_{t,t+1} = \int_t^{t+1} \sigma^2(X_s)ds,$$

which is a stochastic quantity. Realized volatility consistently estimates this quantity (drift not important). Asymptotic mixed normality.

- Also true for **Stochastic volatility models** such as

$$dX_t = \mu_t dt + \sigma_t dW_t$$

$$d\sigma_t = m_t dt + v_t dW_t$$

such as Heston model.

Volatility Measures using only open, close, high and low

- Yahoo, Bloomberg etc all report the daily opening price, closing price, the intraday high price, and the intraday low price: P_O, P_C, P_H, P_L.
- Most authors work with the daily closing price and returns as we have described them have been computed this way.

Definition

A simple measure of volatility is

$$V_t^{HL} = \frac{P_{Ht} - P_{Lt}}{P_{Lt}}.$$

- Actually, can replace the denominator by P_{Ct} for example without much change in the result. This also has an interpretation inside continuous time models.

Definition

The Parkinson (1980) estimator

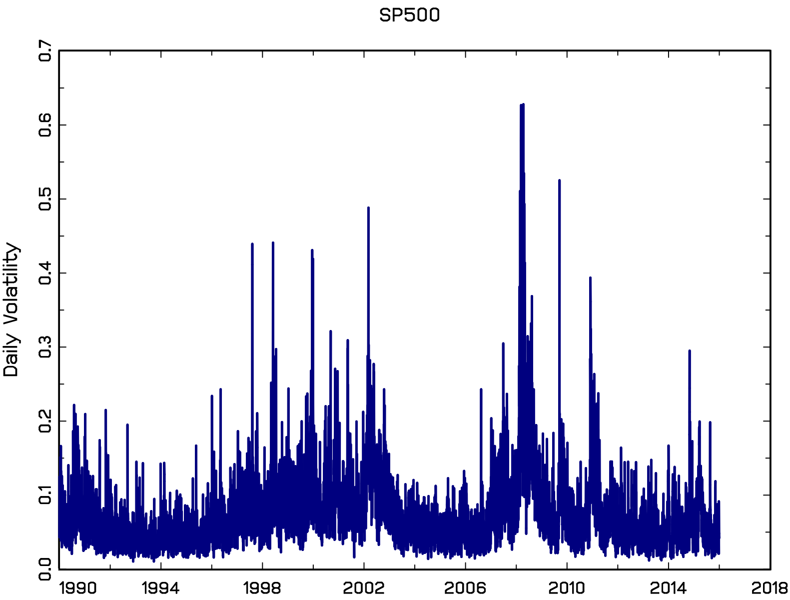
$$V_t^P = \frac{(\log P_{Ht} - \log P_{Lt})^2}{4 \log 2}$$

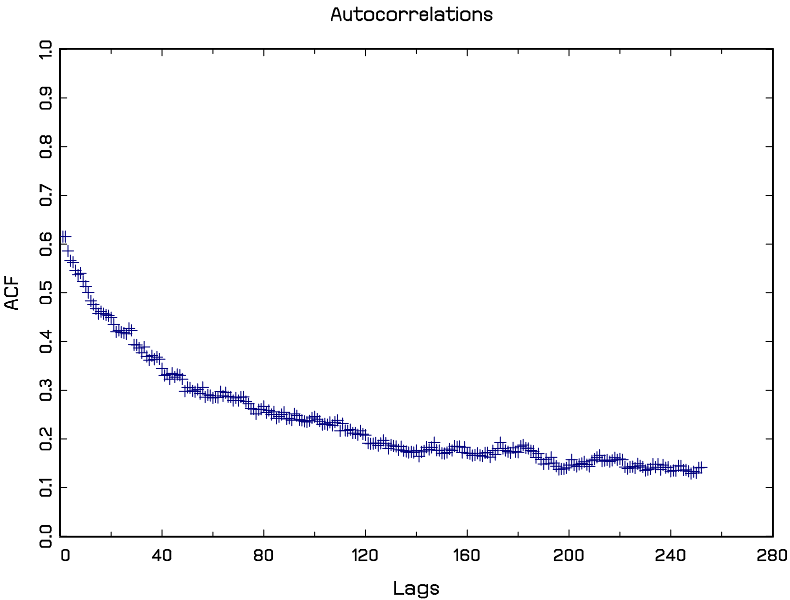
- This consistently estimates the parameter σ^2 of the Brownian motion proces. But it is an inefficient estimator of σ^2 under the model assumption.
- The Garman and Klass (1980), Rogers and Satchell (1991) estimators provide some improvement in efficiency and correct for a drift:

$$V_t^{GK} = 0.5 \left(\ln P_t^H - \ln P_t^L \right)^2 - (2 \ln 2 - 1) \left(\ln P_t^C - \ln P_t^O \right)^2$$

$$V_t^{RS} = (\ln P_t^H - \ln P_t^C)(\ln P_t^H - \ln P_t^O) + (\ln P_t^L - \ln P_t^C)(\ln P_t^L - \ln P_t^O)$$

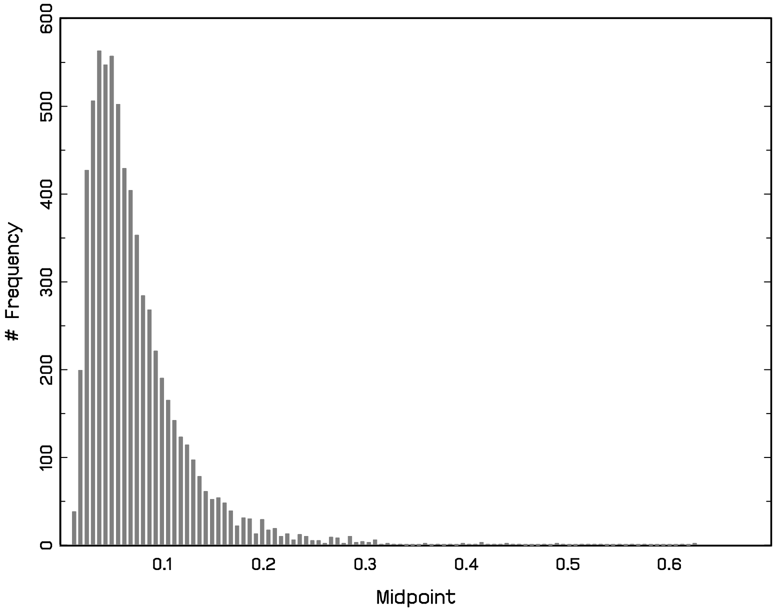
Chou et al. (2009) for a discussion of range based volatility estimators.


SP500
Daily Volatility
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
1990
1994
1998
2002
2006
2010
2014
2018

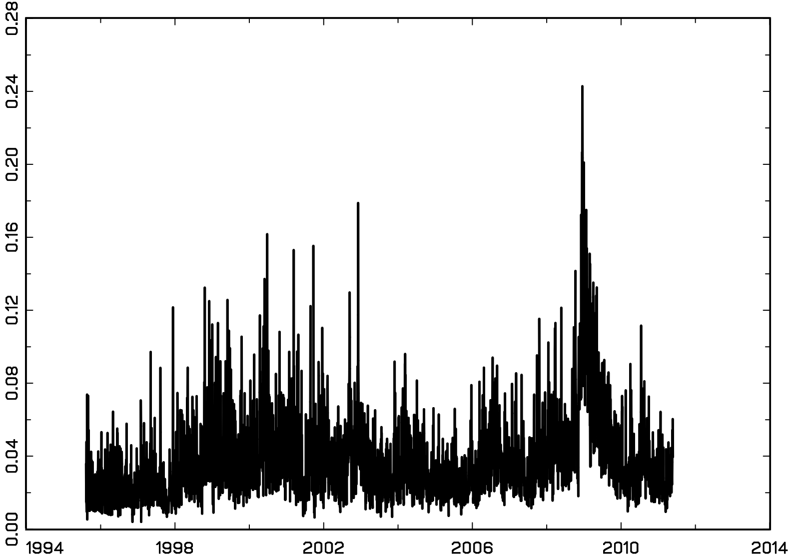


Autocorrelations
ACF
Lags
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
40
80
120
160
200
240
280

Skewed, long right tail



Daily Copper Volatility



FTSE100 Top 20 Most Volatile days since 1984 (- means $P_C < P_O$, + means $P_C > P_O$)

Date	Volatility
19871020	0.131-
19871022	0.115-
20081010	0.112-
19971028	0.096-
20081024	0.096-
20081006	0.094-
20081008	0.094-
20080919	0.093+
20081124	0.090+
20081015	0.084-
19871019	0.081-
20020920	0.080+
20081013	0.076+
20010921	0.076-
20081029	0.075+
20110809	0.074+
20090114	0.074-
20080122	0.074+
20020715	0.071-
20081016	0.070-

S&P500 Top 20 Most Volatile days since 1960

Date	Volatility
19871019	0.257-
19871020	0.123+
20081010	0.107-
20081009	0.106-
20081113	0.104+
20081028	0.101+
20081015	0.100-
20081120	0.097-
20081013	0.094+
20080929	0.093-
19871026	0.092-
20100506	0.090-
20081201	0.089+
19620529	0.089+
19871021	0.087+
20081016	0.087+
20081006	0.085-
20081022	0.085-
20020724	0.081+
19980831	0.080-

- Both markets dominated by 2008 and 1987
- US market a little more volatile than UK
  - perhaps explained by more innovation? perhaps not?
- Circuit breakers now limit the worst case, or perhaps spread it out over several days

Time Series GARCH Models

- Empirically, squared high-frequency returns have strong positive autocorrelation
- Interested in ex ante measures of volatility
- Investors wish to trade-off risk versus return based on current knowledge.

Definition

Engle (1982), Bollerslev (1986) Generalized AutoRegressive Conditional Heteroskedasticity GARCH(1,1) model

$$r_t = \sigma_t \varepsilon_t$$
$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma r_{t-1}^2,$$

where ε_t is i.i.d normal with mean zero and variance one.

- Provided $\omega > 0$ and $\beta, \gamma \geq 0$, then $\sigma_t^2 > 0$ with probability one and
$$\sigma_t^2 = \mathrm{var}(r_t \,|\mathcal{F}_{t-1}),$$
where $\mathcal{F}_{t-1}$ is past information. Proper variance
- Provided
$$\beta + \gamma < 1$$
the process r_t is **weakly stationary** and has finite unconditional variance
$$\sigma^2 = E(\sigma_t^2) = \frac{\omega}{1 - \beta - \gamma}.$$

- Can write

$$\begin{aligned} \sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \gamma r_{t-1}^2 \\ &= \omega + \beta\omega + \gamma r_{t-1}^2 + \beta\gamma r_{t-2}^2 + \beta^2\sigma_{t-2}^2 \\ &= \frac{\omega}{1-\beta} + \gamma \sum_{j=1}^{\infty} \beta^{j-1} r_{t-j}^2. \end{aligned}$$

- σ_t^2 depends on all past squared returns. The weighting β^{j-1} of each lagged squared return declines geometrically fast

Dependence of the process

- We can write the process as an ARMA(1,1) in r_t^2. Specifically, note that

$$\begin{aligned} r_t^2 &= \sigma_t^2 + \sigma_t^2(\varepsilon_t^2 - 1) \\ &= \omega + \beta\sigma_{t-1}^2 + \gamma r_{t-1}^2 + \sigma_t^2(\varepsilon_t^2 - 1) \\ &= \omega + (\beta + \gamma) r_{t-1}^2 + \eta_t + \beta\eta_{t-1}, \end{aligned}$$

  where $\eta_t = r_t^2 - \sigma_t^2 = \sigma_t^2(\varepsilon_t^2 - 1)$ is a mean zero innovation uncorrelated with its past, albeit heteroskedastic, i.e., an *MDS*. This guarantees some dependence in r_t^2.

- We have positive dependence, i.e.,

$$\mathrm{cov}(r_t^2, r_{t-j}^2) > 0$$

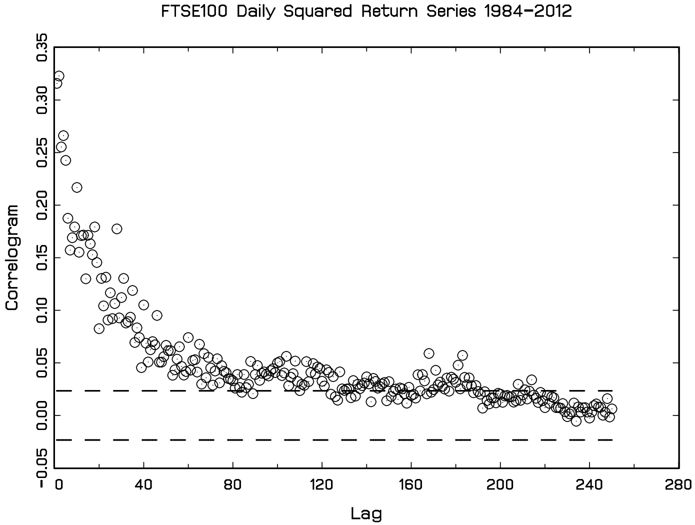
  for all j. This can be observed in the data. Also $\mathrm{cov}(\sigma_t^2, \sigma_{t-j}^2) > 0$

- Persistence of shocks to volatility is measured by $\beta + \gamma$
- In practice, estimated parameters lie close to the boundary of this region i.e., $\beta + \gamma \sim 1$.
- The IGARCH model has

$$\beta + \gamma = 1.$$

In this case, the process r_t is strongly stationary but is not covariance stationary [the unconditional variance is infinite].

We can see that the correlogram of r_t^2 supports this hypothesis for many financial time series



Theorem

The marginal distribution of r_t *will be heavy tailed even if* ε_t *is standard normal. Suppose that* ε_t *is standard normal, then*

$$\frac{E\left(\varepsilon_t^4\right)}{\left(E\left(\varepsilon_t^2\right)\right)^2} = 3.$$

Furthermore, lets suppose that the kurtosis and all fourth order moments are well defined and time invariant (the process r_t *is stationary). Then we have (provided* $\beta + \gamma < 1$*)*

$$\frac{Er_t^4}{(Er_t^2)^2} = \frac{3(1-(\beta+\gamma)^2)}{1-2\gamma^2-(\beta+\gamma)^2} > 3$$

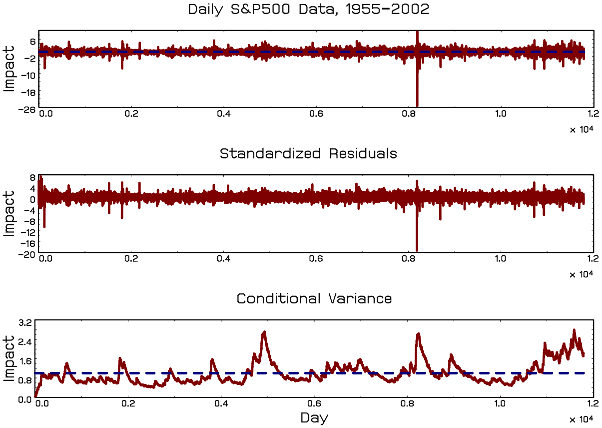
Example S&P500 daily stock index return series from 1955-2002; Eviews.

Parametric Estimation from EVIEWS

	Daily	Weekly	Monthly
ρ_1	0.134457 (0.009531)	0.002871 (0.021709)	0.020311 (0.049513)
ρ_2	-0.027715 (0.009327)	0.032247 (0.022462)	-0.058747 (0.045839)
ω	$6.46E-07$ ($7.28E-08$)	$1.14E-05$ ($2.50E-06$)	0.000103 ($4.63E-05$)
β	0.913948 (0.002366)	0.845708 (0.014920)	0.870139 (0.040540)
γ	0.082395 (0.001757)	0.131007 (0.012950)	0.074298 (0.027528)

Note: Standard errors in parentheses. These estimates are for the raw data series and refer to the AR(2)-GARCH(1,1) model

$$r_t = c + \rho_1 r_{t-1} + \rho_2 r_{t-2} + \overbrace{\varepsilon_t \sigma_t}^{u_t}$$
$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma u_{t-1}^2$$


Daily S&P500 Data, 1955-2002
Impact
6
-2
-10
-18
-26
0.0
0.2
0.4
0.6
0.8
1.0
1.2
× 10^4
Standardized Residuals
Impact
8
4
0
-4
-8
-12
-16
-20
× 10^4
Conditional Variance
Impact
3.2
2.4
1.6
0.8
0.0
Day
× 10^4

Leverage Effect and Asymmetric GARCH

Garch linear-quadratic formulation is theoretically arbitrary, and it misses some empirical patterns:

- Larger responses to negative shocks
- Less than quadratic responses to very large shocks

We consider only the first one. We expect that when $r_{t-1} < 0$ the effect on subsequent volatility is great than when $r_{t-1} > 0$ holding constant the magnitude, i.e., the sign of returns matter

Definition

Leverage hypothesis is that negative returns lower equity price thereby increasing corporate leverage, thereby increasing equity return volatility

- We have $r = |r| \times \mathrm{sign}(r)$. Therefore, we could measure leverage by $\mathrm{cov}(\sigma_t^2, \mathrm{sign}(r_{t-j}))$; equivalently $\mathrm{cov}(\sigma_t^2, r_{t-j})$
- In a pure GARCH model with mean zero returns

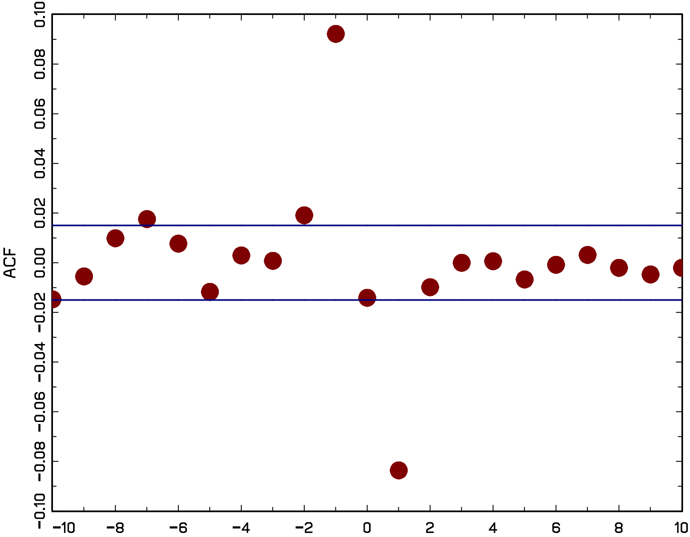
$$\begin{aligned} \mathrm{cov}(\sigma_t^2, r_{t-j}) &= \gamma \sum_{k=1}^{\infty} \beta^{k-1} \mathrm{cov}(r_{t-k}^2, r_{t-j}) \\ &= \gamma \sum_{k=1}^{\infty} \beta^{k-1} E(\sigma_{t-k}^2 \sigma_{t-j}) E(\varepsilon_{t-k}^2 \varepsilon_{t-j}) = 0. \end{aligned}$$

  This is zero because

$$E(\varepsilon_{t-k}^2 \varepsilon_{t-j}) = 0$$

  for all k, j. If $k \neq j$ this is true by independence of ε_s over s, when $j = k$ this is true if ε_t is symmetrically distributed about zero, e.g., normal distribution).
- Therefore, the classic GARCH models rule out leverage effect.

Evidence for leverage effect S&P500 Daily return cross autocovariance $\text{cov}(r_t^2, r_{t-j})$, $j = -10, \ldots, 10.$



- We write

$$x^2 = \overbrace{x^2 1(x > 0)}^{\text{good news}} + \overbrace{x^2 1(x \leq 0)}^{\text{bad news}}$$

- Glosten, Jeganathan and Runkle (1994)

$$\begin{aligned} \sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \gamma r_{t-1}^2 + \delta r_{t-1}^2 1(r_{t-1} < 0) \\ &= \omega + \beta\sigma_{t-1}^2 + \psi_+ r_{t-1}^2 1(r_{t-1} \geq 0) + \psi_- r_{t-1}^2 1(r_{t-1} < 0), \end{aligned}$$

where $\gamma = \psi_+ + \psi_-$ and $\delta = \psi_-$. Equivalent parameterizations.

Parametric Estimation from EVIEWS

	Daily	Weekly	Monthly
ρ_1	0.138788 (0.009524)	0.007065 (0.022000)	0.014661 (0.045131)
ρ_2	−0.01906 (0.009449)	0.051815 (0.022044)	−0.018694 (0.045083)
$\omega(\times 1000)$	0.0000721 (0.0000064)	0.00130 (0.000242)	0.862000 (0.249000)
β	0.920489 (0.002243)	0.850348 (0.015580)	0.442481 (0.176365)
γ	0.034018 (0.002613)	0.047885 (0.013504)	−0.076662 (0.042047)
δ	0.078782 (0.003302)	0.140013 (0.020349)	0.266916 (0.094669)

Note: Standard errors in parentheses. These estimates are for the raw S&P500 data series and refer to the AR(2)-AGARCH(1,1) model

$$r_t = c + \rho_1 r_{t-1} + \rho_2 r_{t-2} + \overbrace{\varepsilon_t \sigma_t}^{u_t}$$
$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \gamma u_{t-1}^2 + \delta u_{t-1}^2 1(u_{t-1} < 0)$$

Computing GARCH Estimates

- The log-likelihood for a general-variant Garch model given normal ε_t has a simple form

$$\ell_T(\omega, \beta, \gamma) = c - \frac{1}{2} \sum_{t=1}^{T} \frac{r_t^2}{\sigma_t^2(\omega, \beta, \gamma)} - \frac{1}{2} \sum_{t=1}^{T} \log \sigma_t^2(\omega, \beta, \gamma)$$

- Use numerical techniques to minimize minus the summed log likelihood of the sample
- The oft-observed near-flatness of the Garch likelihood surface means that T must be large for reliable estimates

Multivariate volatility models

- Multivariate models treat the whole covariance matrix as time-varying. Define

$$\Sigma_t = E(r_t r_t^\top | \mathcal{F}_{t-1}) = \left(\mathrm{cov}_{t-1}(r_{it}, r_{jt})\right)_{i,j},$$

for some $n \times 1$ vector of mean zero series r_t. Bollerslev et al. (1988)

$$h_t = \mathrm{vech}(\Sigma_t) = A + Bh_{t-1} + C\mathrm{vech}(r_{t-1} r_{t-1}^\top),$$

where A is an $n(n+1)/2 \times 1$ vector, while B, C are $n(n+1)/2 \times n(n+1)/2$ matrices.

- The cross-section is naturally large with asset returns data. Naive Garch extension has $\frac{n^2(n+1)^2}{2} + \frac{n(n+1)}{2}$ so with modest $n = 1000$ this requires estimating five hundred billion ($5 * 10^{11}$) parameters!
- Relatively flat Garch log likelihood function requires $\frac{T}{\#parameters}$ to be large for reliable estimation

Parsimony in Multivariate GARCH Models

- Factor GARCH model where the factors f_t are observed (centred and orthogonal) portfolios or macrovariables

$$r_{it} = b_i^\intercal f_t + \varepsilon_{it}$$

$$f_{kt} = \sigma_{k,t} e_{kt}, \ k = 1, \ldots, K$$

$$\sigma_{k,t}^2 = \omega_k + \beta_k \sigma_{k,t-1}^2 + \gamma_k f_{k,t-1}^2,$$

where e_{kt} are iid with mean zero and variance one and mutually independent.

- Then

$$\mathrm{cov}(r_{it}, r_{jt} | \mathcal{F}_{t-1}) = \sigma_{ij,t} = b_i^\intercal \mathrm{diag}\{\sigma_{1,t}^2, \ldots, \sigma_{K,t}^2\} b_j + s_{ij}$$

Strict factor model has $s_{ij} = \mathrm{cov}(\varepsilon_{it}, \varepsilon_{jt}) = 0$ if $i \neq j$

- If the factors are observed, then estimate b_i by time series regression. Estimate GARCH(1,1) parameters using the factor time series one by one

CCC Model

Definition

For $i, j = 1, \ldots, n$

$$r_{it} = \sigma_{i,t}\varepsilon_{it}$$

$$\sigma_{i,t}^2 = \omega_{ii} + \beta_i \sigma_{i,t-1}^2 + \gamma_i r_{i,t-1}^2$$

$$\varepsilon_{it} \text{ iid with } E\varepsilon_{it} = 0,\ E\varepsilon_{it}^2 = 1,\ E\varepsilon_{it}\varepsilon_{jt} = \rho_{ij},$$

$$\sigma_{ij,t} = \rho_{ij}(\sigma_{i,t}^2 \sigma_{j,t}^2)^{1/2}$$

- Estimate univariate Garch models using maximum likelihood, then calculate the sample correlation matrix of the standardised outcomes
- DCC model of Engle and Sheppard (2002), see corrected version cDCC papers.ssrn.com/sol3/papers.cfm?abstract_id=1507743

GARCH in Mean

- In intertemporal general equilibrium, the risk premium associated with equities might increase when volatility increases
- The Garch-M model assumes that the risk premium is linear in known function of volatility

$$\mu_t = E(r_t|\mathcal{F}_{t-1}) = \alpha_0 + \alpha_1 g(\sigma_t^2), \quad g(x) = x, \sqrt{x}, \ln x$$

$$\sigma_t^2 = \mathrm{var}(r_t|\mathcal{F}_{t-1}) = \sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \gamma(r_{t-1} - \mu_{t-1})^2$$

Parametric Estimation from EVIEWS			
	Daily	Weekly	Monthly
α	0.081504 (0.029699)	0.121757 (0.076905)	0.415873 (0.327167)
ω	$6.49E-07$ ($7.48E-08$)	$1.13E-05$ ($2.53E-06$)	0.000125 (0.072803)
β	0.916160 (0.002356)	0.846601 (0.014707)	0.858988 (0.044015)
γ	0.079801 (0.001737)	0.130387 (0.012697)	0.072803 (0.027614)

Note: Standard errors in parentheses. These estimates are for the raw S&P500 data series and refer to the GARCH(1,1) in mean model

$$r_t = c + \alpha\sigma_t + \varepsilon_t\sigma_t$$
$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \gamma u_{t-1}^2$$

Nonstationarity

- Recently, a criticism of GARCH processes has come to the fore, namely their usual assumption of stationarity.
- By taking $\beta + \gamma \geq 1$ one can have nonstationary processes, but at the cost of non-existence of unconditional variance.
- Instead, maybe the coefficients change over time, thus

$$\sigma_t^2 = \omega_t + \beta_t \sigma_{t-1}^2 + \gamma_t r_{t-1}^2$$

with $\beta_t + \gamma_t < 1$

	50-60	60-70	70-80	80-90	90-00	00-09
c	0.0213	0.0019	-0.0106	0.0111	0.0043	-0.0234
ρ_1	0.1584	0.2229	0.2429	0.0635	0.0603	-0.0814
ρ_2	-0.0977	-0.0288	-0.0563	-0.0033	0.0120	-0.0445
ω	0.0425	0.0166	0.0039	0.0605	0.0132	0.0121
β	0.8330	0.8086	0.9543	0.8620	0.9230	0.9416
γ	0.0584	0.0574	0.0073	0.0362	0.0016	-0.0179
δ	0.0692	0.2031	0.0691	0.0980	0.1264	0.1278
R^2	0.0165	0.0320	0.0515	0.0025	0.0000	0.0171
$mper$	0.0607	0.1941	0.1866	0.0602	0.0723	-0.1259
$vper$	0.9260	0.9676	0.9962	0.9472	0.9878	0.9876
μ_{year}	0.1199	0.0714	0.0346	0.0939	0.0768	0.0161
σ_{year}	0.1144	0.1080	0.1520	0.1616	0.1571	0.1492

$$r_t = c + \rho_1 r_{t-1} + \rho_2 r_{t-2} + \varepsilon_t \sigma_t$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma u_{t-1}^2 + \delta u_{t-1}^2 1(u_{t-1} < 0)$$

Volatility and EMH

- We have seen substantial evidence that volatility of asset returns varies over time in a way that can be partially predicted. Does this violate market efficiency?
- The answer is no unless a trading strategy could be designed that would use this information in the options markets to identify under- and over-valued options. If options markets are efficient, option prices should incorporate the best volatility forecasts at all points in time

Schwert (1989,2010)

- He examines monthly US volatility computed as sum of squared daily returns for 1885-1987 and a regression model approach like GARCH for 1857-1985.
- Main findings
  - The average level of volatility is higher during (NBER dated) recessions
  - The level of volatility during the great Depression was very high
  - The effect of financial leverage on volatility is small
  - There is weak evidence that macroeconomic volatility can help to predict financial asset volatility and stronger evidence for the reverse prediction
  - The number of trading days in the month is positively related to stock volatility (Trading days per year NYSE 252, LSE 255 (but 24 Dec is half day))
  - Share trading volume growth is positively related to stock volatility

French and Roll (1986) Volatility over weekend and holidays

- Calendar time hypothesis: Variance is proportional to calendar time
- Trading time hypothesis: Variance is proportional only to the trading time

Typical trading day may be 8 hours long out of 24 hours (say 8-4).
Weekend, Friday close to Monday open contains 64 hours.
Suppose that hourly stock returns satisfy

$$r_t \sim \mu_h, \sigma_h^2$$

Then daily returns (open to close) satisfy

$$r_t \sim 8\mu_h, 8\sigma_h^2$$

Monday open from friday close returns satisfy

$$r_t \sim \begin{cases} 64\mu_h, 64\sigma_h^2 & \text{Calendar time} \\ 0 & \text{Trading Time} \end{cases}$$

- They calculate variance as the average of squared returns over stocks and over the relevant period

$$varperhour_{monday} = \frac{1}{n_{monday}} \sum_{mondays} \frac{\left(p_{mclose} - p_{mopen}\right)^2}{8}$$

$$varperhour_{weekend} = \frac{1}{n_{weekend}} \sum_{weekends} \frac{\left(p_{mopen} - p_{fclose}\right)^2}{64}$$

- They find that per hour return variance is 70 times larger during a trading hour than during a weekend hour

Is this because:

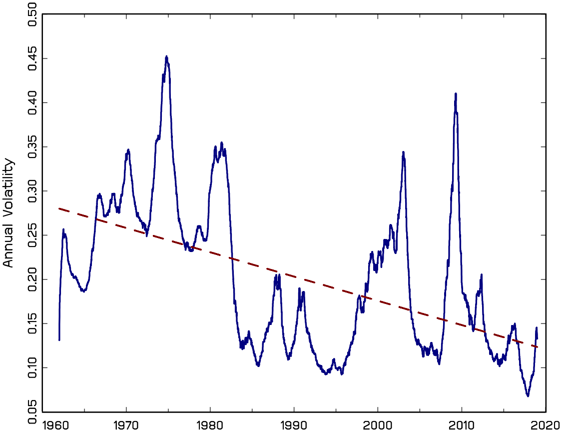
1. Volatility is caused by public information which is more likely to arrive during normal business hours
2. Volatility is caused by private information which affects prices when informed investors trade
3. Volatility is caused by pricing errors that occur during trading

They find:

- There are some pricing errors (evidenced from autocorrelation) due to microstructure and misspricing issues but most is caused by information release
- To distinguish between public and private information (explanations 1 and 2) they use the fact that in 1968, NYSE was closed every wednesday because of "paperwork crisis", but otherwise was a regular business day. Explanation 2 is their main story.

Is volatility trending upwards?

Shows rolling window annual median of daily RS volatility estimator for S&P500 with trend line



Valid cases: 14261 Dependent variable: Y
Missing cases: 0 Deletion method: None
Total SS: 102.907 Degrees of freedom: 14259
R-squared: 0.283 Rbar-squared: 0.283
Residual SS: 73.767 Std error of est: 0.072
F(1,14259): 5632.752 Probability of F: 0.000
Standard Prob Standardized Cor with
Variable Estimate Error t-value $>|t|$ Estimate Dep Var

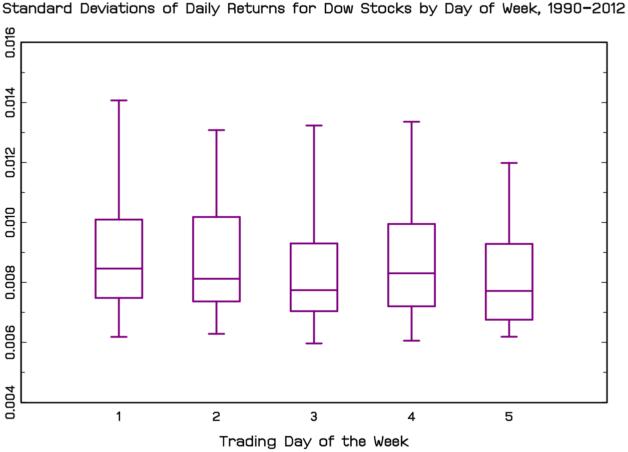
CONSTANT 0.280094 0.001205 232.509159 0.000 — —
X1 -0.000011 0.000000 -75.051662 0.000 -0.532137 -0.532137

Conclusions

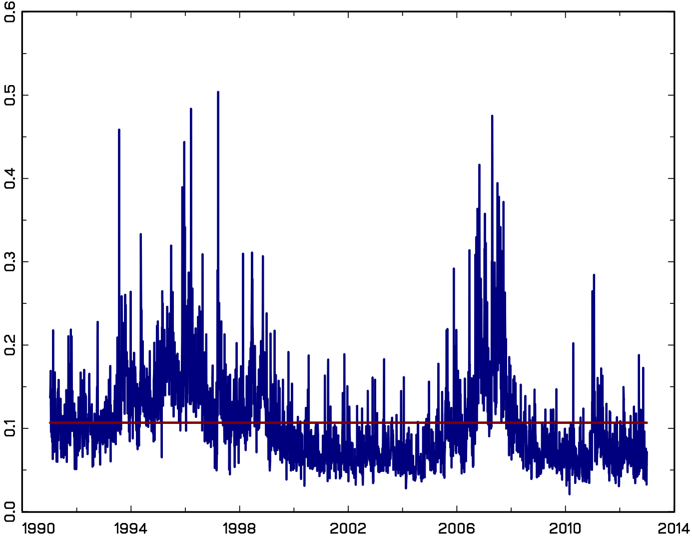
- Different ways of measuring volatility, ex post and ex ante
- Applications
  - Risk management. Value at risk
  - Asset pricing
  - Permanent/transitory volatility

Annualized std, and idiosyncratic std

	σ	σ_ε		σ	σ_ε
Alcoa Inc.	0.2151	0.1676	JP Morgan	0.0991	0.0806
AmEx	0.1851	0.1160	Coke	0.2234	0.1591
Boeing	0.2350	0.1717	McD	0.1491	0.1367
Bank of America	0.1540	0.1191	MMM	0.1458	0.1217
Caterpillar	0.1595	0.1168	Merck	0.1972	0.1811
Cisco Systems	0.1103	0.0923	MSFT	0.2092	0.1818
Chevron	0.2151	0.1695	Pfizer	0.2057	0.1901
du Pont	0.1504	0.1251	P & Gamble	0.1045	0.0872
Walt Disney	0.1408	0.0980	AT&T	0.1327	0.1058
General Electric	0.2267	0.2007	Travelers	0.1703	0.1402
Home Depot	0.2200	0.1862	United Health	0.2132	0.1968
HP	0.1937	0.1614	United Tech	0.1852	0.1578
IBM	0.2014	0.1686	Verizon	0.1280	0.1034
Intel	0.1318	0.0978	Wall Mart	0.1268	0.1022
Johnson[2]	0.2268	0.1880	Exxon Mobil	0.1470	0.1218


Standard Deviations of Daily Returns for Dow Stocks by Day of Week, 1990-2012
0.004
0.006
0.008
0.010
0.012
0.014
0.016
1
2
3
4
5
Trading Day of the Week

Cross-sectional volatility $\widehat{\sigma}_t = \sqrt{252 * \sum_{i=1}^{n} (r_{it} - \overline{r}_t)^2 / (n-1)}$ Relatively low around 2000 but very high 2008



Rolling Monthly version