F500 Empirical Finance
Lecture 3: Empirical Market Microstructure

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Outline

1. Microstructure Explanations/Models for serial correlation in stock returns
   1. Stale Prices or infrequent trading. Stale prices are due to infrequent trading (now one usually talks about stale quotes)
   2. Price discreteness: prices are quoted in multiples of the minimum price increment (tick size)
   3. Bid/Ask Bounce

2. Explanations for the existence and magnitude of Bid/Ask (Offer) spread

3. Strategic Trading and the measurement of liquidity

4. Recent Developments

Reading: Linton (2019, Chapter 5), Campbell, Lo and MacKinlay (1997, Chapter 3)
Stale Prices

- CLM model of non trading. There is underlying true price, but it is only observed when a transaction occurs.
- In practice, stocks trade with different frequency, from Apple at one end (many times a millisecond) to "penny stocks" that may only trade once a week. Many empirical questions are concerned with the cross-section of returns, and nonsynchronous trading is a big problem in high frequency data.
Suppose that the "underlying" price sequence $P_t$ and hence returns $r_t$ for each $t = 1, 2, \ldots$

$$r_t \text{ i.i.d. } (\mu, \sigma^2)$$

Observed (or transaction time) return depends upon the last two prices at which transactions took place, hence it is the sum of intervening subperiod calendar returns.

Suppose that

$$\delta_t = \begin{cases} 
1 \text{ (no trade)} & \text{with probability } \pi \\
0 \text{ (trade)} & \text{with probability } 1 - \pi 
\end{cases}$$

The frequency of trading is controlled by the parameter $\pi$. Calendar time model (from lecture 1) is special case with deterministic $\delta_t$. 
- Observed price \( p_t^O \)

\[
p_t^O = \begin{cases} 
  p_t & \text{if there is a trade at } t \\
  p_{t-1}^O & \text{otherwise}
\end{cases}
\]

- If \( \delta_t = 1 \), then

\[
  r_t^O = p_t^O - p_{t-1}^O = p_{t-1}^O - p_{t-1}^O = 0.
\]

- If \( \delta_t = 0 \), then

\[
  r_t^O = p_t - p_{t-1}^O,
\]

where \( p_{t-1}^O \) may depend on \( \delta_{t-1}, \delta_{t-2}, \ldots \)

- The key quantity is how stale is the price \( p_{t-1}^O \), which depends on how many periods have \( \delta_{t-j} = 1 \)
Definition

The duration of non trading, denoted $d_t$, is given by the integer $k$ for which $\delta_{t-k} = \delta_t = 0$ but $\delta_s = 1$ for $s \in (t - k, t)$ and $d_t = 0$ if $t$ and $t - 1$ have trades.

- The random variable $d_t \in \{0, 1, 2, \ldots\}$, and depends on the past sequence of $\{\delta_t\}$.
- It has a dynamic evolution

$$d_{t+1} = \begin{cases} 
  d_t + 1 & \text{with probability } \pi \\
  0 & \text{with probability } 1 - \pi.
\end{cases}$$

This is a stationary Markov stochastic process that evolves over time.
The observed (logarithmic) return is \( r_t^O = p_t^O - p_{t-1}^O \).

\[
r_t^O = \begin{cases} 
0 & \text{with prob } \pi \text{ (no trade today)} \\
 r_t & \text{with prob } (1 - \pi)^2 \text{ (trade today and yday)} \\
 r_t + r_{t-1} & \text{with prob } (1 - \pi)^2 \pi \text{ (trade today and day b4 yday)} \\
 & \vdots \\
 & \vdots 
\end{cases}
\]

\[
r_t^O = \begin{cases} 
0 & \text{with prob } \pi \\
 \sum_{k=0}^{d_t} r_{t-k} & \text{with prob } 1 - \pi 
\end{cases}
\]
<table>
<thead>
<tr>
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<th>$p_1$</th>
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<td>no trade? $\delta$</td>
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<td>observed price</td>
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<td>true return</td>
<td>$r_2$</td>
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<tr>
<td>obs return</td>
<td>$r_2$</td>
<td>$r_3$</td>
<td>0</td>
<td>0</td>
<td>$r_4 + r_5 + r_6$</td>
<td>$r_7$</td>
<td>$r_8$</td>
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<tr>
<td>duration $d$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
The marginal distribution of $d_t$ is Geometric (type 2) with $p = 1 - \pi$. That is,

$$
d_t = \begin{cases} 
0 & 1 - \pi \\
1 & (1 - \pi)\pi \\
2 & (1 - \pi)\pi^2 \\
\vdots & \vdots
\end{cases}
$$

Therefore, we can show

$$
Ed_t = (1 - \pi)\pi + 2(1 - \pi)\pi^2 + \cdots
$$

$$
= (1 - \pi)\pi \sum_{j=0}^{\infty} (j + 1)\pi^j = \frac{\pi}{1 - \pi}
$$

$$
\text{var}(d_t) = \frac{\pi}{(1 - \pi)^2}
$$
Main Implications for single stock

- We can show that:

\[
E(r_t^O) = \mu \\
\text{var}(r_t^O) = \sigma^2 + \frac{2\pi}{1 - \pi}\mu^2 \\
\text{cov}(r_t^O, r_{t+n}^O) = -\mu^2 \pi^n.
\]

- This is consistent with observed returns following an ARMA(1,1) process such that

\[
r_t^O - \mu = \pi(r_{t-1}^O - \mu) + \eta_t + \theta \eta_{t-1},
\]

where \(\eta_t\) is iid mean zero with variance \(\sigma_{\eta}^2\) and \(\theta\) is such that \(|\theta| < 1\).
Cross Covariances

- We consider the bivariate case with

\[ \text{cov}(r_{it}, r_{jt}) = \sigma_{ij}. \]

We find empirically that \( \sigma_{ij} \gg 0 \) for most pairs of stocks. Each security has non trading probability \( \pi_i \).

- Can show that for \( i \neq j, n \geq 0 \)

\[ \gamma_{ij}(n) = \text{cov}(r_{it}^O, r_{jt+n}^O) = \frac{(1 - \pi_i)(1 - \pi_j)}{1 - \pi_i \pi_j} \sigma_{ij} \pi_j^n \]

- Predicts that cross-autocorrelations are the same sign as \( \sigma_{ij} \), ie usually positive. Note \( \frac{(1 - \pi_i)(1 - \pi_j)}{1 - \pi_i \pi_j} \leq 1 \)

- These formulae explain some of patterns in autocorrelations and cross autocorrelations in individual stocks reported in CLM
Relative cross-covariances depend only upon the trading frequencies of the securities

\[
\frac{\gamma_{ij}(n)}{\gamma_{ji}(n)} = \frac{\text{cov}(r_{it}^O, r_{jt+n}^O)}{\text{cov}(r_{jt}^O, r_{it+n}^O)} = \left( \frac{\pi_j}{\pi_i} \right)^n.
\]

Note that the more liquid stock (small $\pi$) tends to lead more strongly and the less liquid stock tends to lag more strongly, but both lead and lag effects are present unless $\pi = 1$ for one stock.
The non trading model gives some useful insights, especially with regard to non synchronous trading. However:

- CLM model is nominally about daily data (in 1980s this made sense) Nowadays all S&P500 stocks, say, trade every day many times now, so perhaps more relevant for small stocks or corporate bonds or intraday data
- Problem for intraday application is have to choose some unit of time: In a trading day (LSE is 0800-1630 and NYSE is 0930-1600) there may be 400 minutes, 25000 seconds, 25000000 milliseconds etc.
- Observable quoted prices contain information beyond that contained in the most recent trade and are updated more frequently than trades. Use the miquote.
- Magnitudes of effects too small to explain autocorrelation properties (CLM)
- Only relevant now for intraday transactions data. In that case, \( \pi \) is surely not fixed within a day and depends on past trades and prices as well as time of day. Engle ACD model and many others push this forward.
Discreteness

- Quantity and Prices are discrete (minimum price increment, minimum quantity). In 1997 when CLM was published, the minimum price increment or tick size in the US was 1/8th of a dollar.

- In the United States, for any stock over $1 in price level the tick size is currently one cent (although there are exceptions - Berkshire Hathaway A priced at $134,060.00 only takes $1 moves).

- US is currently debating "subpenny pricing" whereby tick size might be reduced to 0.1 of a cent. In FX, tick size can be 0.0001 or smaller.

- In UK (and most countries except USA), tick size varies across stocks in bands according to the price level and market capitalization (generally speaking more liquid stocks have small tick sizes). Tick size has been subject to regulatory debate.
Bid ask bounce: transaction prices oscillate between bid price and ask price.
Consequence of discreteness is negative autocorrelation in observed price changes.

Why? The logic is a bit like a non trading model except it also works even if there is no drift. If the rounding bites often, then many returns are zero and positive return is most likely followed by zero return and negative return is most likely followed by zero return.

Suppose that

\[ P_t = P_{t-1} + \varepsilon_t \]

where \( \varepsilon_t \) is \( N(0, \sigma^2) \), \( P_0 = 10 \) and \( \sigma = 0.1 \), and let

\[ P^R_t = \text{round}(P_t, 1/8) \]

which means the closest point to \( P_t \) on the grid \( \{j/8, j = 1, 2, \ldots\} \). Then \( P^R_t - P^R_{t-1} \) is negatively autocorrelated at first lag.
Round off errors are approximately uniform. We show histogram of $16 \times (P_t - P_t^R)$.
Bid, Ask and Transaction Prices

Dealer market: Dealer/market maker quote bid and ask prices either as needed or displayed publicly (think travelex). Take it or leave it. Knows the flow of orders. In some case is a monopolist and has unique access to the order flow information; in other cases competitive dealers.

Electronic Order Book Anyone can enter buy or sell orders. Transparent display of demand and supply. Competition between liquidity providers.
The theoretical literature is mostly concerned with a stylized version of the dealer market where:

- Market participants can be sorted into **outsiders** (everyone except the market maker) and **insiders** (the market maker).
- The market maker sets the bid ask spread. He hopes to earn the spread, i.e., to buy at the bid and sell at the ask.
- If she can make these two trades at exactly the same time, this is a money printing machine. But she can’t and it is generally a risky business as we see.
Roll Model

- Roll model: What is the consequence of a (fixed) Bid-ask spread for price changes?
- Assume that fundamental price $P^*$ is a random walk
  \[ P_t^* = P_{t-1}^* + \varepsilon_t. \]

- Dealer sets ask price $P_t = P_t^* + \frac{s}{2}$ and bid price $P_t = P_t^* - \frac{s}{2}$, where $s$ is the spread. Buy and sell orders arrive randomly unrelated to (independent of) fundamental price. Transaction price is
  \[ P_t = P_t^* + Q_t \frac{s}{2}, \]
  where $Q_t$ is a trade direction indicator, $+1$ for buy and $-1$ if customer is selling. A special case of the Fads model!
- It follows that
  \[ \Delta P_t = \Delta P_t^* + (Q_t - Q_{t-1}) \frac{s}{2}. \]
Assume that $Q_t$ is iid with equal probability of +1 and -1 and unrelated to $P_t^*$. 

Since by definition of a random walk, $P_t^*$ has zero autocovariance and by Roll’s assumption $Q_t$ is unrelated to $P_t^*$

$$\text{cov} [\Delta P_{t-1}, \Delta P_t] = -\frac{s^2}{4}$$

This is *Bid Ask Bounce* (BAB) - presence of bid ask spread induces negative first order autocorrelation in transaction price.

Note that $\Delta P$ is an MA(1) process with

$$\text{cov} [\Delta P_{t-j}, \Delta P_t] = 0, \quad j \geq 2$$

Multivariate case with cross-correlated order flow and heterogeneous spreads leads to Vector MA(1) process

The midquote $M_t = P_t^*$ is serially uncorrelated.
This gives an estimation equation for the spread, $s$:

$$s = 2 \sqrt{- \text{cov} [\Delta P_{t-1}, \Delta P_t]}$$

This is sometimes used to estimate the spread (when this is not directly observed) from transaction data.

And the noise of the random walk (fundamental volatility) is

$$\sigma^2_\varepsilon = \text{var} [\Delta P_{t-1}] + 2 \text{cov} [\Delta P_{t-1}, \Delta P_t].$$

If use returns instead of price changes get percentage spread.

Empirically problematic. Many covariances positive. A number of refinements to this measure that address this issue have been suggested, see for example Hasbrouck (2006).
Efficient Price Variance For Daily Dow Individual Stocks, 1990–2012

Capitalization Ranked

0 4 8 12 16 20 24 28 32

0.00002 0.00006 0.00010 0.00014
What determines the bid-ask spread?

- Inventory models: The bid-ask spread compensates the market maker for the risk of ruin through inventory explosion.
- Information models: Bid-ask spread is determined by adverse selection costs.
- Combinations allow both adverse selection and inventory costs.
Inventory Models

- Important role of market makers: provide opportunity to trade at all times ("immediacy") to traders
- Market makers (MM) absorb temporary imbalances in order flow and will hold inventory of assets. Inventory may deviate from desired inventory position (in the long run, zero) due to changes in demand and supply
- The bid ask spread $S$ is independent of the current inventory level and

$$S \simeq A + \sigma^2 \theta T q$$

where: $A$ is a fixed measure of the monopoly power of the MM, $q$ is the trade size for which the quote is relevant, $\sigma^2$ is volatility of stock price, $\theta$ is a measure of the risk aversion of the MM, and $T$ is his time horizon
Information Models

- Adverse selection is an important issue for market makers/dealers. Harder to price discriminate as insurance companies do to mitigate risks.
- The bid-ask spread is compensation for the market maker’s adverse selection costs.
A simple sequential trade model


- Value $V$ is chosen from the distribution

$$V = \begin{cases} V_H & \text{with prob } 1 - \delta \\ V_L & \text{with prob } \delta \end{cases}$$

- Type of investor is chosen each period from

$$T = \begin{cases} I & \text{with prob } \mu \\ U & \text{with prob } 1 - \mu \end{cases}$$

- Strategies:
  - If informed (I), buy if value is high $V_H$ and sell if value is low $V_L$
  - If uninformed (U), buy or sell with probability 1/2
  - Dealer sets $B, A$ to make zero profits (this is forced on her by competition or regulation). Observes noisy signal, the order flow
Each period there is a buy or sell order. Let $Q$ be the indicator of order direction, i.e., $Q = +1$ if the order is a buy order and $Q = -1$ is the order is a sell order.

The order flow is described by the following probability table:

<table>
<thead>
<tr>
<th></th>
<th>$Q = +1$</th>
<th>$Q = +1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V = V_L$</td>
<td>$\frac{1}{2} (1 - \mu)$</td>
<td>$\frac{1}{2} (1 + \mu)$</td>
</tr>
<tr>
<td>$V = V_H$</td>
<td>$\frac{1}{2} (1 + \mu)$</td>
<td>$\frac{1}{2} (1 - \mu)$</td>
</tr>
</tbody>
</table>

where for example the probability of receiving a buy order when the asset has low value is $\Pr(Q = +1|V = V_L) = (1 - \mu) / 2$. 
Dealer reasons: if receive a buy order how would I update my valuation?

By Bayes rule, she would calculate posterior distribution given $Q$

$$
\Pr(V = V_L | Q = +1) = \frac{\Pr(Q = +1 | V = V_L) \Pr(V = V_L)}{\Pr(Q = +1)} = \frac{\frac{1}{2} (1 - \mu) \times \delta}{(1 + \mu (1 - 2\delta))/2}
$$

Then

$$
\Pr(V = V_H | Q = +1) = 1 - \Pr(V = V_L | Q = +1).
$$

Likewise compute $\Pr(V = V_L | Q = -1)$ and $\Pr(V = V_H | Q = -1)$. 
Zero expected profit condition (side by side) implies that:

\[ A = E(V|Q = +1) = V_L \frac{\delta(1-\mu)}{1+\mu(1-2\delta)} + V_H \frac{(1-\delta)(1+\mu)}{1+\mu(1-2\delta)} \]

\[ B = E(V|Q = -1) = \frac{V_L \delta(1+\mu) + V_H (1-\delta)(1-\mu)}{1-\mu(1-2\delta)} \]

Therefore, Bid-ask spread is

\[ s = A - B = \frac{4(1-\delta)\delta\mu}{1-(1-2\delta)^2\mu^2} (V_H - V_L) \]
When \( \delta = 1/2 \), the bid-ask spread is

\[
 s = A - B = \mu(V_H - V_L)
\]

- Bid ask spread is wider when there are more informed investors
- Bid ask spread is wider when \( V_H - V_L \) is larger (could interpret this as volatility or uncertainty over final value)
- Dealer gains from uninformed traders and loses to informed ones.
- Midpoint is \( M = (V_L + V_H)/2 = E(V) \) the unconditional expectation of value.
Now consider this process evolving over time (with new traders every period but constant values $V_H, V_L$).

Dealer observes a price/order history at any time $t - 1$, for example \( \{Q_1 = 1, Q_2 = -1, Q_3 = -1, \ldots, Q_{t-1} = 1\} \), denote this by $\mathcal{F}_{t-1}$. Orders are serially correlated - informed investors always trade in the same direction. He updates his valuation of the security given $\mathcal{F}_{t-1}$.

Prior is now replaced by the posterior distribution from the last round

\[ \Pr(V = V_L | \mathcal{F}_{t-1}) , \] whence obtain the posterior

\[
\Pr(V = V_L | Q_t = 1, \mathcal{F}_{t-1}) = \frac{\Pr(Q_t = 1 | V = V_L, \mathcal{F}_{t-1}) \Pr(V = V_L | \mathcal{F}_{t-1})}{\Pr(Q_t = 1 | \mathcal{F}_{t-1})}
\]

Trades (or rather order flow) have price impact because they lead to an adjustment in the posterior.

Sets the next period ask and bid price by the same zero profit condition

\[
A_t = E(V | Q_t = 1, \mathcal{F}_{t-1}) ; \quad B_t = E(V | Q_t = -1, \mathcal{F}_{t-1})
\]
The transaction price is

\[
P_t = \begin{cases} 
    A_t & \text{if } Q_t = +1 \\
    B_t & \text{if } Q_t = -1
\end{cases} = M_t + Q_t \frac{s_t}{2} = E(V|\mathcal{F}_t)
\]

This is a martingale with respect to \( \mathcal{F}_t \) (weak/semi-strong form EMH) because

\[
E[P_{t+1} | \mathcal{F}_t] = E[E[V | \mathcal{F}_{t+1}] | \mathcal{F}_t] = E[V | \mathcal{F}_t] = P_t
\]

by the law of iterated expectation, \( \mathcal{F}_t \subset \mathcal{F}_{t+1} \).

- This says that microstructure effects do not necessarily lead to serial correlation.
- Information is improving over time:
  - Market maker can consistently estimate the true value from a long sequence of orders
  - Spreads get narrower over time
  - Prices converge to the true value
Simulated examples. Chose \( V = V_H = 1 \) with \( V_L = 0 \)

1. \( \mu = 0.5, \delta = 0.5 \)
2. \( \mu = 0.1, \delta = 0.5 \)

Shows as time goes by

- in panel A, Ask price, Bid price, transaction price (red squares)
- in panel B, bid ask spread
- in panel C, inventory of dealer
Glosten (1987). Considers more general model where bid ask spread is determined by a combination of: adverse selection, order processing costs, inventory costs, and monopoly power. In this case, transaction price is no longer a martingale.

Order processing costs etc are temporary in effect, whereas information has permanent effect on prices.

Glosten and Harris (1988). Develop empirical model containing adverse selection and inventory components. They find that the adverse selection component is significant and is related to relevant stock characteristics.
- More realistic models (Easley and O’Hara (1987, 1992) (VPIN) allows
  - Informed investors do not know precisely the value of stock but observe private signals (information events).
  - Not all trading days have information events
  - Trades can be of different sizes and price impact depends on trade size
  - Participants may or may not trade so the time between trades is variable and this may have information content
Strategic trade models

- Value of security $v \sim N(\mu_v, \sigma_v^2)$. Informed trader knows $v$ and submits demand $x(v)$ to maximize his expected profit.
- Noise traders submit order flow $u \sim N(0, \sigma_u^2)$.
- Risk neutral Market maker observes total demand $y = x + u$ and then sets a price $p$ to make zero profits in expectation.
- Uses Bayes theorem to try to estimate $E(x|y)$. 
Equilibrium is a pricing rule of the market maker $p(y)$ and informed trader demand function $x(v)$ that are mutually consistent.

Can show that

$$p(y) = \mu_v + \frac{1}{2} \sqrt{\frac{\sigma_v^2}{\sigma_u^2}} y$$

$$x(v) = -\mu_v \sqrt{\frac{\sigma_u^2}{\sigma_v^2}} + \sqrt{\frac{\sigma_u^2}{\sigma_v^2}} v$$

Expected profit of the informed trader

$$\frac{(v - \mu_v)^2}{2} \sqrt{\frac{\sigma_u^2}{\sigma_v^2}} \geq 0$$

at the expense of the noise traders. MM breaks even in the long run. Half of the insiders information is impounded in price

$$\text{var}[v|y] = \sigma_v^2 / 2$$

Continuous time version of model: informed trader chops up their order and feeds into the market.
Definition

Kyle’s Lambda

\[ p(y) = \mu_v + \lambda y, \quad \lambda = \frac{1}{2} \sqrt{\frac{\sigma_v^2}{\sigma_u^2}} \]

- This is the amount that the market maker raises the price when the total order flow \( y \) goes up by 1 unit (\( \lambda = dp/\text{dy} \)). Hence, the amount of order flow necessary to raise the price by $1 equals \( 1/\lambda \), which is a measure of the “depth” of the market or market “liquidity.”

- The higher is the proportion of noise trading to the value of insider information, the deeper or more liquid is the market. Intuitively, the more noise traders relative to the value of insider information, the less the market maker needs to adjust the price in response to a given order, since the likelihood of the order being that of a noise trader, rather than an insider, is greater.
There are a number of ways of measuring liquidity from low frequency data, see Goyenko et. al. (2009, JFE) for a review. Amihud (2002) develops a liquidity measure that captures the daily price response associated with one unit of trading volume.

**Definition**

Let

\[ Am_t = \frac{1}{N_t} \sum_{j=1}^{N_t} \ell_{t_j}, \quad \ell_{t_j} = \frac{|R_{t_j}|}{V_{t_j}} , \]

where \( V \) is the trading volume and \( R \) returns. Here, we should average daily \( \ell_{t_j} \) over a longer period like a week or a month.

Large values of this measure indicate an illiquid market where small amounts of volume can generate big price moves. It is considered a good proxy for the theoretically founded Kyle’s price impact coefficient.
This shows the raw unaveraged $\ell_{t_j}$ for daily data.
The limit order book

<table>
<thead>
<tr>
<th>Bid</th>
<th>Ask</th>
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</thead>
<tbody>
<tr>
<td>Price</td>
<td>Quantity</td>
</tr>
<tr>
<td>15.71</td>
<td>2000</td>
</tr>
<tr>
<td>15.70</td>
<td>4500</td>
</tr>
<tr>
<td>15.69</td>
<td>5000</td>
</tr>
<tr>
<td>15.68</td>
<td>10000</td>
</tr>
<tr>
<td>15.67</td>
<td>15000</td>
</tr>
</tbody>
</table>

At any one time, this is available to (some) participants. Market orders or aggressive limit orders (that cross the spread) will execute against "the book". Market buy order for 15000: 7000@15.72, 3000@15.73, 4000@15.74, 1000@15.75 (walking down the book). Volume weighted average price (VWAP)

\[
VWAP = \frac{7}{15} \times 15.72 + \frac{3}{15} \times 15.73 + \frac{4}{15} \times 15.74 + \frac{1}{15} \times 15.75 = 15.729
\]
New order book. Bid ask spread is 0.04.

<table>
<thead>
<tr>
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<tr>
<td>15.69</td>
<td>5000</td>
</tr>
<tr>
<td>15.68</td>
<td>10000</td>
</tr>
<tr>
<td>15.67</td>
<td>15000</td>
</tr>
</tbody>
</table>

Until replenished by new limit orders
This is (mechanical) **market impact**
If the market buy order were for quantity 2000, then the spread would not change. The order executes by **Price Time Priority** - the first order at the price gets executed first and so on
Real time limit order book: [http://www.batstrading.co.uk/](http://www.batstrading.co.uk/)
Posting limit orders (supplying liquidity) gives options to trade to other traders. Provides a service to other traders. Needs to be compensated.

The value of those options can be calculated from Black and Scholes (1973, JPE) call option price

\[
C(S, K, \tau, r_f, \sigma) = S \cdot \Phi(d_+) - K \cdot e^{-r_f \cdot \tau} \cdot \Phi(d_-)
\]

\[
d_\pm = \frac{\log\frac{S}{K} + \left(r_f \pm \frac{\sigma^2}{2}\right) \cdot \tau}{\sigma \cdot \sqrt{\tau}}
\]

and \(\Phi\) is the standard normal cdf. At the money, \(S = K\), as \(\tau \to 0\)

\[
C\left(S, K, \tau, r_f, \sigma\right) = \frac{1}{\sqrt{2\pi}} S \cdot \sigma \sqrt{\tau} + O(\tau).
\]

There is a positive albeit small value in an order that only sits for small time.

If there are many orders, then value = large \times small
Markets import new information quickly.
Appendix

Since \( \{\delta_t\} \) is independent of \( \{r_t\} \) we can condition on the sequence \( \{\delta_t\} \) and use the law of iterated expectation

\[
Er_t^0 = (1 - \pi)E \left[ E \left( \sum_{k=0}^{d_t} r_{t-k} \mid d_t \right) \right] = (1 - \pi)\mu E(d_t + 1) = \mu
\]

\[
E[(r_t^0)^2] = (1 - \pi)E \left[ E \left( \left( \sum_{k=0}^{d_t} r_{t-k} \right)^2 \mid d_t \right) \right]
\]

\[
= (1 - \pi) \left[ E(d_t + 1)Er_t^2 + E((d_t + 1)d_t)E^2 r_t \right]
\]

\[
= (1 - \pi) \left[ E(d_t + 1)(\sigma^2 + \mu^2) + E((d_t + 1)d_t)\mu^2 \right]
\]

\[
= (1 - \pi) \left[ \frac{1}{1 - \pi}(\sigma^2 + \mu^2) + \frac{2\pi}{(1 - \pi)^2}\mu^2 \right]
\]
We calculate $E[r_t^O r_{t+1}^O]$ by first conditioning on the sequence $\{\delta_t\}$. Unless $\delta_{t+1} = \delta_t = 0$, we have $r_t^O r_{t+1}^O = 0$, so we only need consider the case that $\delta_{t+1} = \delta_t = 0$, which has probability $(1 - \pi)^2$. In this case, $d_{t+1} = 0$. It follows that

$$E[r_t^O r_{t+1}^O] = (1 - \pi)^2 E \left( r_{t+1} \sum_{i=0}^{d_t} r_{t-i} \right)$$

$$= (1 - \pi)^2 \mu^2 E(d_t + 1)$$

$$= \mu^2 (1 - \pi)^2 \left( \frac{\pi}{1 - \pi} + 1 \right)$$

$$= \mu^2 (1 - \pi).$$

Therefore, as in CLM (1997, p89)

$$\text{cov}(r_t^O, r_{t+1}^O) = -\pi \mu^2.$$