F500 Problem Set 3

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1. Slowly varying expected returns. Suppose that
$$r_{t+1} = x_t + \varepsilon_{t+1},$$
where
$$x_{t+1} = \mu + \phi x_t + \xi_{t+1}, \quad -1 < \phi < 1$$
and ε_t, ξ_s are mutually independent for all t, s and are individually iid with mean zero and variances σ_ε^2 and σ_ξ^2. Calculate $Er_t, \mathrm{var}(r_t)$, $E_t\left[r_{t+1}\right]$, and $\mathrm{var}_t\left[r_{t+1}\right]$. Compute the unconditional autocorrelation function
$$\rho(k) = \frac{\mathrm{cov}(r_t, r_{t-k})}{\mathrm{var}(r_t)}$$
Is this consistent with the empirical evidence regarding autocorrelation of return series?

2. Blanchard and Watson (1982) model. Suppose that
$$B_{t+1} = \begin{cases} \frac{1+R}{\pi} B_t + \eta_{t+1} & \text{with probability } \pi \\ \eta_{t+1} & \text{with probability } 1-\pi \end{cases}$$
where η_t is iid with mean zero and variance one. What are the properties of the bubble process? What is $E_t B_{t+1}$? What is $\mathrm{var}_t B_{t+1}$? What is the chance that the bubble lasts for more than 5 periods? Suppose that we observed prices satisfy
$$P_t = P_t^* + B_t$$
$$P_t^* = P_{t-1}^* + u_t$$
where u_t is normally distributed with mean zero and variance one. Suppose also that you have a method for identifying observations for which the bubble is in operation (i.e., the first regime of B is in operation). How would you test for the presence of the bubble?

3. Suppose that
$$r_t = \rho r_{t-1} + \varepsilon_t.$$
We can write
$$(1 - \rho L) r_t = \varepsilon_t$$
Consider the process r_t^2. We have
$$r_t^2 = \rho^2 r_{t-1}^2 + \varepsilon_t^2 + 2\rho r_{t-1} \varepsilon_t$$
so that
$$(1 - \rho^2 L) r_t^2 = \sigma_\varepsilon^2 + u_t,$$
where $u_t = \varepsilon_t^2 - E\varepsilon_t^2 + 2\rho r_{t-1}\varepsilon_t$ is a martingale difference sequence. It follows that
$$r_t^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2} + \sum_{j=0}^{\infty} \vartheta^j u_t,$$
where $\vartheta = \rho^2$. It follows that
$$\text{corr}(r_t^2, r_{t-j}^2) = \vartheta^j = \rho^{2j}.$$
It follows that for this process the autocorrelation of the squares dies out faster than the autocorrelation of the original series. This is not supported by the data. Suppose that returns follow a moving average process
$$r_t = \varepsilon_t + \theta \varepsilon_{t-1},$$
where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. Recall that
$$\text{corr}(r_t, r_{t-j}) = \begin{cases} \frac{\theta}{(1+\theta^2)} & \text{if } j = 1 \\ 0 & \text{else.} \end{cases}$$
Suppose that returns are normally distributed, ie $\text{var}(\varepsilon_t^2) = 2\sigma_\varepsilon^4$. Then show that
$$\text{corr}(r_t^2, r_{t-j}^2) = \begin{cases} \frac{\theta^2}{(1+\theta^2)^2} & \text{if } j = 1 \\ 0 & \text{else.} \end{cases}$$

4. Suppose that you have a time series of daily returns on a stock i that is traded in a different time zone from stock j. Specifically, the trading day for i is the first 1/3 of the day, and the trading day for j is the second third of the day. The final third of the day contains no trading. We observe the closing prices for each asset on their respective "trading days", which we denote by $P_{i1}, P_{i4}, \ldots,$ and $P_{j2}, P_{j5}, \ldots$ We want to calculate the contemporaneous return

covariance. We assume that each stock has iid return and that the contemporaneous covariance between return on stock i and stock j is γ, that is,

$$\text{cov}(p_{it} - p_{i,t-1}, p_{it+s} - p_{i,t+s-1}) = 0$$

$$\text{cov}(p_{it} - p_{i,t-1}, p_{jt} - p_{j,t-1}) = \gamma.$$

Then show that

$$\text{cov}(p_{i4} - p_{i1}, p_{j5} - p_{j2}) = \text{cov}(p_{i4} - p_{i2}, p_{j4} - p_{j2}) = 2\text{cov}(p_{i2} - p_{i1}, p_{j2} - p_{j1})$$

So we should use stale price adjustment

$$\frac{2}{3}\text{cov}(r_{it}, r_{jt}) + \frac{1}{3}\text{cov}(r_{it}, r_{j,t-1})$$

5. Consider the GARCH (1,1) model:

$$r_t = h_t^{1/2}\eta_t$$

$$h_t = \omega + \beta h_{t-1} + \gamma r_{t-1}^2$$

where h_t is the conditional variance of time t returns. For notational convenience assume that the asset's expected return equals zero. a) Explain the restrictions on the parameters of the GARCH(1,1) model required to ensure that the long-run unconditional variance exists, b) Describe the unconditional variance in terms of these parameters, c) Discuss how the values of the parameters affect the persistence of the response of dynamic volatility to a return shock.