F500 Problem Set 2

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1. Suppose that X and Y are mean μ random variables with var$X = \sigma_X^2$ and var$Y = \sigma_Y^2$ and suppose that $\mathrm{cov}(X, Y) = \sigma_{XY} = \sigma_X \sigma_Y \rho_{XY}$ with $|\rho_{XY}| \leq 1$. Invest a fraction ω of your wealth in X and $1 - \omega$ in Y, called portfolio $P(\omega)$. Show that:

   (a) For all $\omega \in [0, 1]$

   $$\mathrm{var}(P(\omega)) \leq \max\{\sigma_X^2, \sigma_Y^2\}$$

   (b) The optimal ω satisfies

   $$\omega_{opt} = \frac{\sigma_Y^2 - \sigma_X \sigma_Y \rho_{XY}}{\mathrm{var}(X - Y)}$$

   (provided $\mathrm{var}(X - Y) > 0$), and for this value

   $$\mathrm{var}(P(\omega_{opt})) \leq \min\{\sigma_X^2, \sigma_Y^2\}$$

   (c) Under what conditions would ω be negative?

2. Consider the market model

   $$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it},$$

   where ε_{it} are iid and mean and independent of R_{mt}, which are themselves iid with finite variance σ_m^2. Show that the covariance matrix of returns satisfies

   $$\Omega = \sigma_m^2 \beta \beta^\intercal + D$$

3. Consider the market model

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it},$$

where ε_{it} are iid and mean and independent of R_{mt}, which are themselves iid with finite variance σ_m^2. Now suppose that you regress Z_{it} on a constant and Z_{mt}, where $Z_{it} = R_{it} - R_{ft}$ and $Z_{mt} = R_{mt} - R_{ft}$, where R_{ft} is the risk free rate. What would you expect to happen to the parameter estimates?

4. What role does the assumption of normality play in testing the Capital Asset Pricing Model? What is the evidence regarding normality in stock returns? If stock returns are not normal and indeed have heavy tailed distributions with some extreme outliers, what are the properties of the standard normal-based tests of this hypothesis?

5. Volatility. Suppose that daily returns are normally distributed with mean μ and variance σ^2. Suppose you have a sample of intraday returns $r_1, \ldots, r_n$ where the day is taken as the unit interval $[0, 1]$ and returns r_i are the returns from t_{i-1} to t_i, where $t_i \in [0, 1]$. Suppose that $r_i \sim N(\mu(t_i - t_{i-1}), \sigma^2(t_i - t_{i-1}))$. Investigate the properties of the realized volatility

$$RV_n = \sum_{i=1}^{n} r_i^2$$