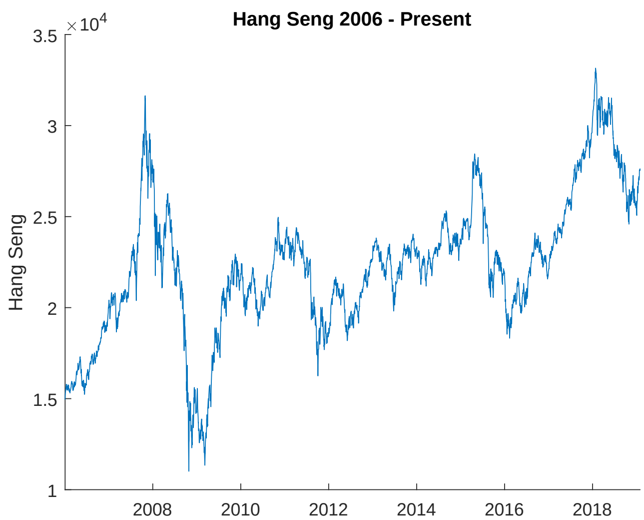
F500 Problem Set 1 - Solutions

Dr Tom Auld*

February 11, 2019

Question 1

I chose the Hang Seng index from Hong Kong, its largest constituent; China Construction Bank and a much smaller small cap, FDG Electric Vehicles. Statistics are derived from the last 12 years so as to allow comparison across the 3 assets:



Price Returns Statistics:

Mean 0.031
StdDev 1.525
Kurt 12.603
Skew 0.248
JBstat 12421

Log Returns Statistics:

Mean 0.019
StdDev 1.524
Kurt 12.061
Skew -0.008
JBstat 11030

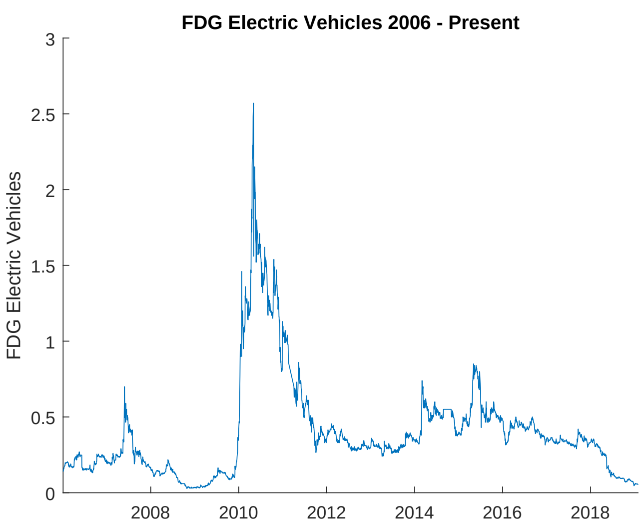
*Comments and corrections to tja20@cam.ac.uk

Price Returns Statistics:

Mean 0.051
StdDev 2.121
Kurt 14.244
Skew 0.836
JBstat 17382

Log Returns Statistics:

Mean 0.029
StdDev 2.107
Kurt 12.085
Skew 0.473
JBstat 11223



Price Returns Statistics:

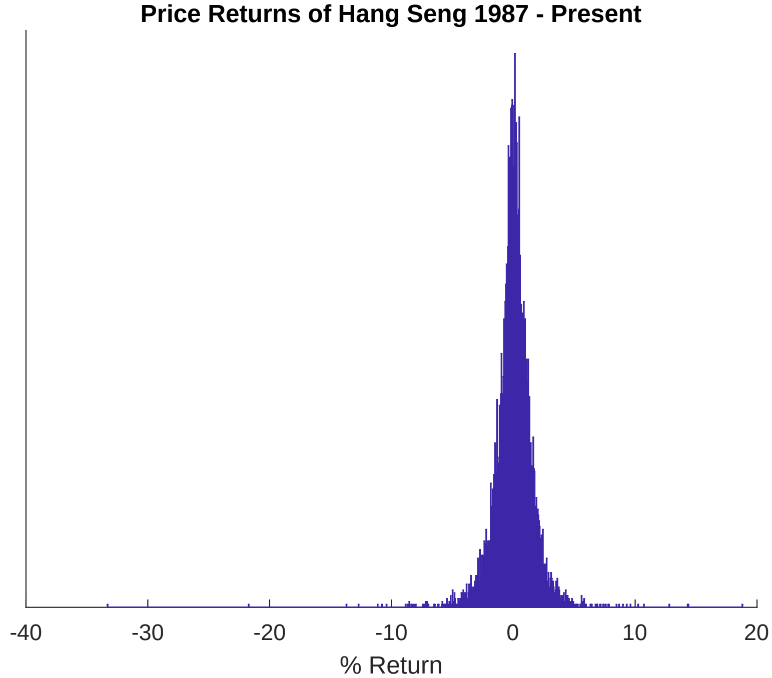
Mean 0.087
StdDev 4.990
Kurt 24.129
Skew 2.235
JBstat 62051

Log Returns Statistics:

Mean -0.031
StdDev 4.809
Kurt 16.788
Skew 1.108
JBstat 25946

In price return the series all have positive drift but interestingly the small stock has average negative log return - this suggests there are large positive values in price returns that contribute to a positive average but have less of a contribution to the average of the log returns (log is a concave function). This is apparent in the price chart. The variance of the small stock is higher than the large cap which is in turn higher still than the index. This is expected from what we learnt in lectures. However the Skew is positive which is unexpected - asset prices typically exhibit negative skew, called leverage, explained by *holders* of an asset selling on large down moves, exacerbating the sell-off. Finally, all returns series exhibit large excess kurtosis which is as expected. I have included the Jarque Bera statistic. Under the null of normality, this has an asymptotic χ^2_2 distribution. Clearly all series are hugely significantly different from the normal distribution.

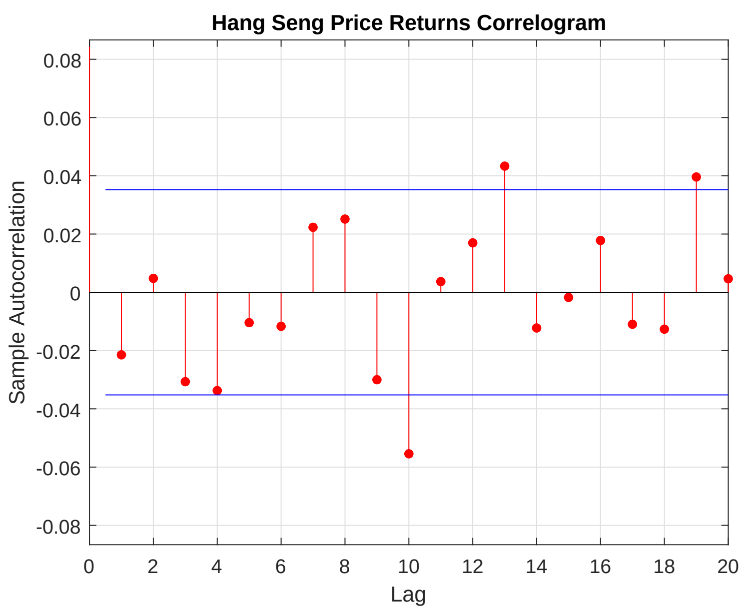
The histogram of returns for the Hang Seng from 1987 is shown below. The skew is negative (-1.22) over this period in line with the stylized facts of stock returns. The skew and excess kurtosis are clearly apparent:



b)

Correlogram

First compute the autocorrelations and compare with the Bartlett intervals:



At first glance 3 autocorrelations appear significant. Further the Ljung-Box Q statistic is 61.5 for 20 lags, rejecting the null. However both the Bartlett intervals and Ljung-Box test assume IID returns (RW1).

An Aside....Testing The Efficient Markets Hypothesis

RW1 $\epsilon_t \sim IID; \quad E(\epsilon_t) = 0$

RW2 $\epsilon_t \sim independent\,over\,time; \quad E(\epsilon_t) = 0$

RW2.5 Martingale Property: $E[\epsilon_{t+1}|\epsilon_t, \epsilon_{t-1}, \ldots] = 0 \quad \Rightarrow E[\epsilon_{t+1} g(\epsilon_t, \epsilon_{t-1}, \ldots)] = 0$ by LIE

RW3 $\forall k > 0, \; COV(\epsilon_t, \epsilon_{t-k}) = 0$

RW2.5 is the **Natural** definition on EMH.

RW1 $\Rightarrow$RW2 $\Rightarrow$ RW2.5 $\Rightarrow$ RW3 so $\overline{RW3} \Rightarrow \overline{RW2.5} \Rightarrow \overline{RW2} \Rightarrow \overline{RW1}$

Testing RW1

RW1 $\Rightarrow$

$$Q = T \sum_{j=1}^{P} \hat{\rho_j}^2 \sim \chi^2_P \qquad \text{Box-Pierce Q Statistic}$$

RW1 $\Rightarrow$

$$Q = T(T+2) \sum_{j=1}^{P} \frac{\hat{\rho_j}^2}{T-j} \sim \chi^2_P \qquad \text{Box-Ljung Q Statistic}$$

Latter has better finite sample properties. Proceed by calculating Q and reject RW1 at level α when $Q > \chi^2_P(\alpha)$

RW1 $\Rightarrow$

$$\sqrt{T}\hat{\rho_k} \Longrightarrow N(0,1) \quad \forall k$$

Test by comparing $\hat{\rho_k}$ with the Bartlett Intervals:

$$\left[-\frac{z_{\alpha/2}}{\sqrt{T}}, \frac{z_{\alpha/2}}{\sqrt{T}} \right]$$

There are also tests of "average" correlation $\bar{\rho}(k)$ across a portfolio of correlated assets under RW1 as well variance ratio tests.

Testing RW2.5

Testing RW1 does not reject the **natural** form of the EMH; RW2.5. We need to correct for het-roskedasticity and non-linearity in higher moments (eg as in a GARCH model).

RW2.5 $\Rightarrow$

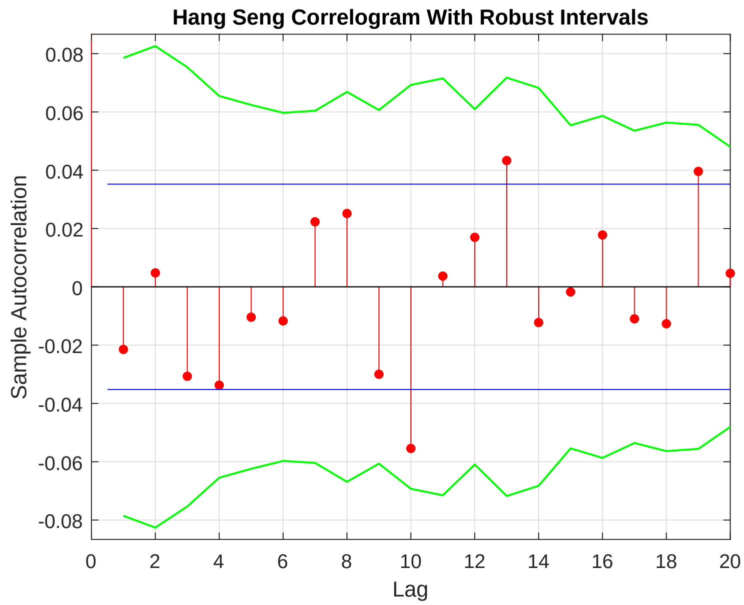
$$\sqrt{T}\hat{\rho_k} \Longrightarrow N\left(0, \frac{E(X_t^2 X_{t-k}^2)}{E(X_t^2)^2}\right) \quad \forall k$$

Conduct test via adjusted Bartlett interval **on de-meaned series** (robust errors):

$$\left[-\frac{z_{\alpha/2}}{\sqrt{T}} \times \sqrt{\frac{\sum_t X_t^2 X_{t-k}^2}{\left(\sum_t X_t^2\right)^2}}, \frac{z_{\alpha/2}}{\sqrt{T}} \times \sqrt{\frac{\sum_t X_t^2 X_{t-k}^2}{\left(\sum_t X_t^2\right)^2}} \right]$$

The difference can be shown to be increasing in the kurtosis of the returns and proportional to the autocorrelation at lag k of the squared return series. Similar adjustments exist for variance ratio tests under RW2.5.

Comparing with the adjusted intervals tells a different story:



The significance disappears! What does this say about the autocorrelation of the squared returns?

Regression Test

We can conduct a regression test where we regress the next period return on some number of lags of that return. The regression appears highly significant, even using robust errors (as below), highlighting the difference between regression and autocorrleation tests. The regression coefficient is based on a *joint* prediction whereas the correlation coefficient is based on a univariate model. The joint model takes into account the dependence of the different lags (which will be non-zero if the model is not trivial, because if a lag predicts the return, then an older lag predicts a newer lag). Note that when explanatory variables are independent, the regression coefficient is the correlation coefficient. The significance of the regression may come as a surprise. However, the coefficients are tiny and of questionable economic significance - for instance for a 1% move in the previous return one should fade the move by 2.4bp (bp = basis point = 0.01%). The R^2 is also tiny and the reported significance is likely down to the large data set used, rather than any meaningful predictability.

```
Linear regression model (robust fit):
    y ~ 1 + x1 + x2 + x3 + x4 + x5

Estimated Coefficients:
                    Estimate          SE          tStat       pValue
                   __________    __________    ________    __________

    (Intercept)    0.00066912    0.00013967      4.7908    1.6917e-06
    x1              -0.024335     0.0085434     -2.8484     0.0044049
    x2              0.0045593     0.0085407     0.53383       0.59347
    x3               0.023911     0.0085333       2.802     0.0050903
    x4             -0.0055994      0.008541    -0.65559       0.51211
    x5              0.0063506     0.0085433     0.74335       0.45729


Number of observations: 7921, Error degrees of freedom: 7915
Root Mean Squared Error: 0.0124
R-squared: 0.0026,  Adjusted R-Squared 0.00197
F-statistic vs. constant model: 4.12, p-value = 0.000965
```

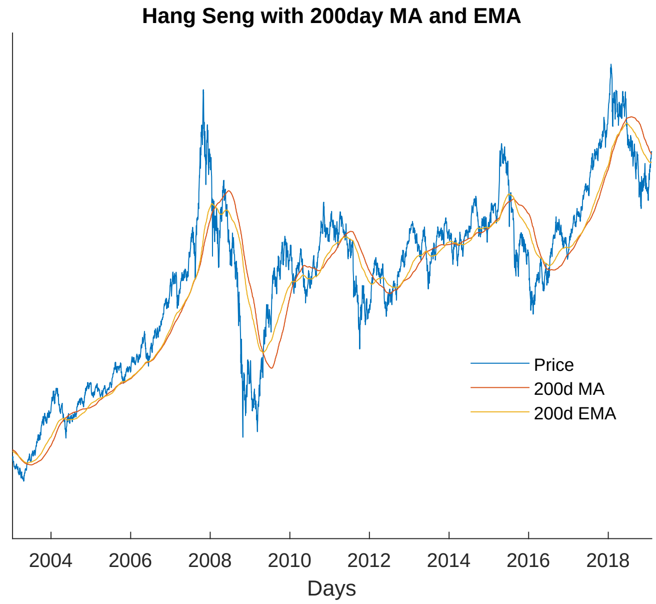
c)

Computing returns using log price ratios makes no difference to the ACF, it looks identical. This is not surprising as $log(1+x) = x + O(x^2) \Rightarrow r_t = R_t + O(R_t)$, so log and price returns are the same to first order. Typically daily returns are under 2% magnitude so any second order contribution is around $2\%^2$ or 0.0004. Of course we do see the odd huge outlier (such as -33% on Black Monday for the Hang Seng!) and the non-linearity does change the moments of the distribution as we saw earlier.

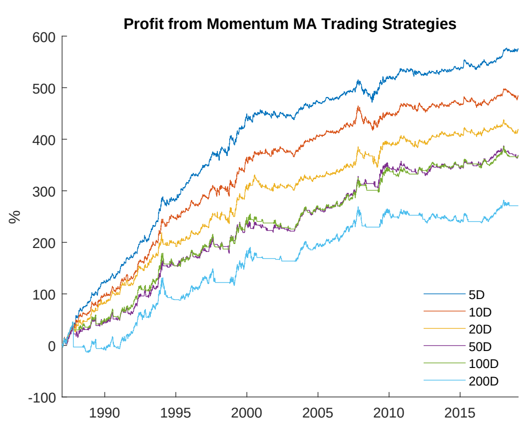
Question 2

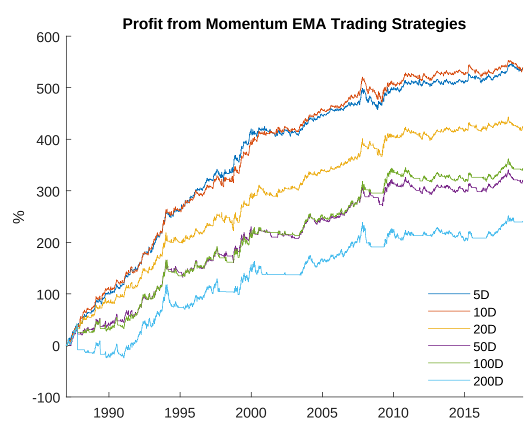
Contrarian Trading Strategies rely on the idea that the value of an asset will overreact in the short term (perhaps due to behavioural factors) but then correct and "revert to the mean". Momentum strategies simply buy when the stock is going up, ie when it is above its mean, and sell on a corresponding negative signal. The strategies outlined in this question, ignoring trading costs, are exact opposites. If the stock closes below the relevant MA the contrarian strategy buys and earns the next day return. The momentum trade buys and earns the opposite return.

A simple glance at the moving averages (below) suggest momentum is more likely to work, as when the price is above the moving average the stock tends to be rising.

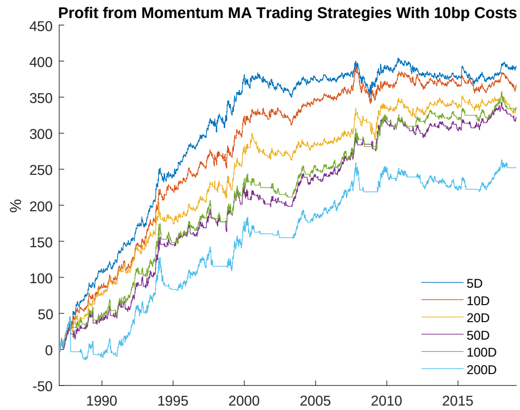


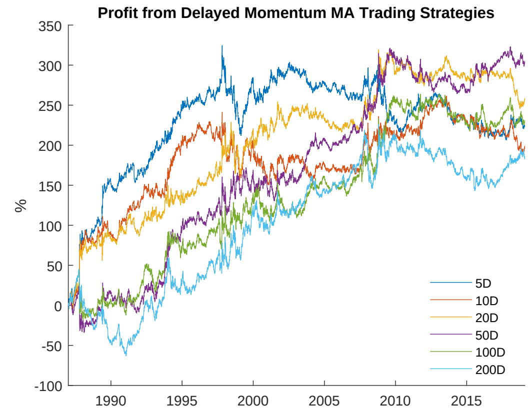
The cumulative returns of the momentum trading strategy appear below (There is no need to show the contrarian returns as they are simply minus those of the above):





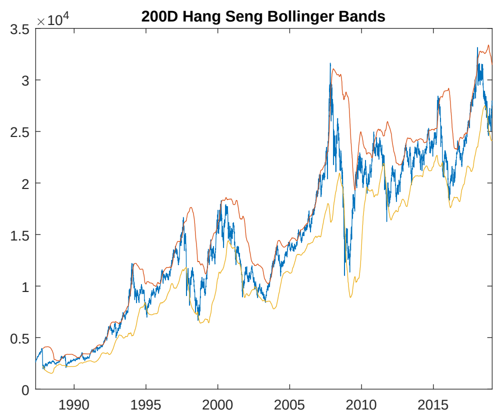
These strategies appear fantastic. However, they ignore trading cost and slippage. Further, they assume that if today's close crosses the moving average one can trade at that close, which is actually impossible (it is more plausible for a contrarian strategy if there is a closing auction - why?). Below results are shown for the simple moving average in the case where there are transaction costs of 10bp, and also when instead of trading on today's close we trade on tomorrows close (with no costs). Clearly the returns are much worse, particularly since 2000 and for the realistic "delayed strategy".

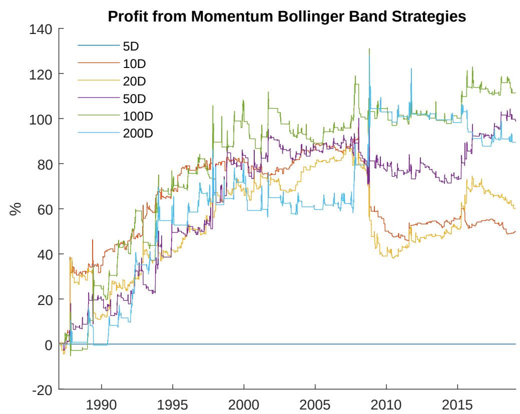




Question 3

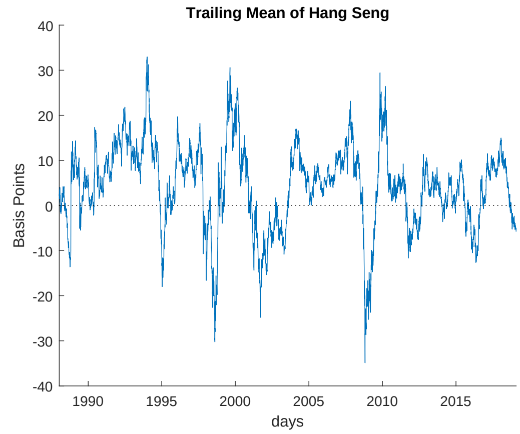
We repeat the analysis from the previous question for the bollinger band momentum strategy. Again there is no need to plot the opposite, contrarian strategy. Returns are calculated ignoring costs and slippage (trading on "today's" close). Note that the absolute returns are lower as these algorithms have no position for much of the time due to the bollinger band providing a "safety net". Costs would likely be reduced significantly, particularly for the shorter lengths. (They are zero for the 5 day moving average as it never trades!).

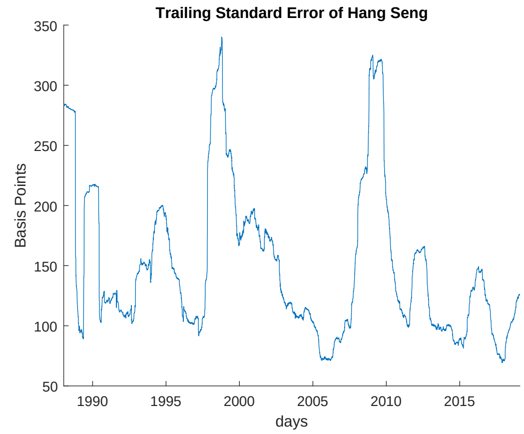


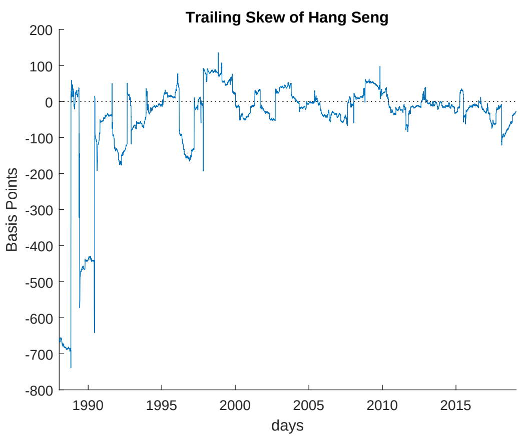


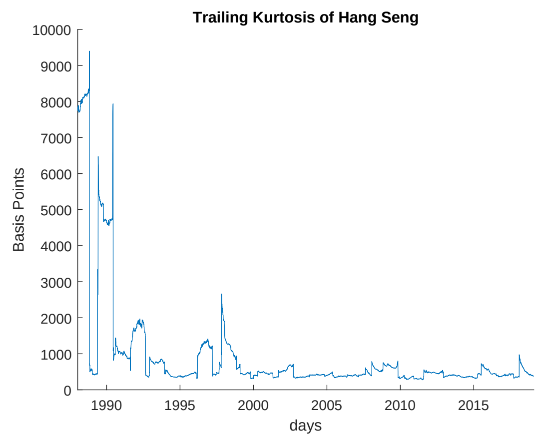
Question 4

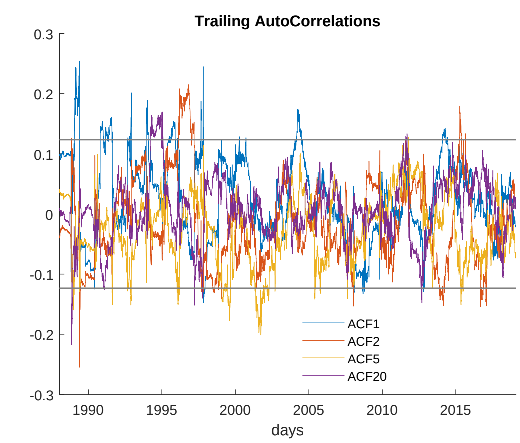
Rolling annual sample mean, standard error, skewness, kurtosis, selected autocorrelations for the Hang Seng shown below and the full price chart for comparison.

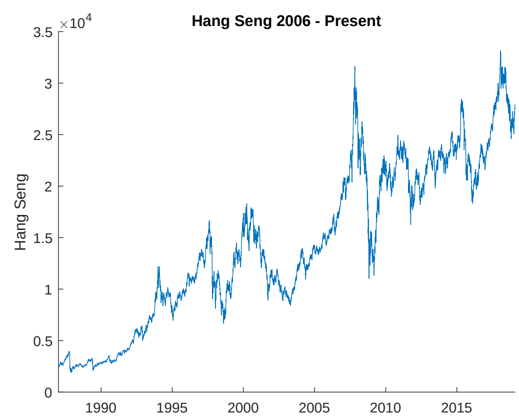












Interesting features of the trailing charts are:

- The trailing mean follows rising and falling markets.
- The trailing standard error has clusters in it. This is indicative of persistent volatility which is a characteristic of financial time series. For interest the spike around 1998 is due to the Russian

Financial Crisis when LTCM nearly blew up the financial system - the FED organised a bailout of \$3.625 billion by the fund's major creditors. There are also spikes around the dotcom bubble in 2001 and the 2008 GFC is also clearly evident.

- The trailing third and fourth standardised moments are affected greatly in the early years by some huge outliers which include Black Monday in 1987 (-33%).
- The trailing autocorrelations do not appear statistically different from noise.

Question 5

Although not explicit in the question assume there are $k-1$ zeros separating reported trades so that the observed and real returns have the same number of data points. We observe j batches of $k-1$ zero returns followed by the sum of all the preceding true returns that were missed. We are asked to consider the difference in the *sample* mean, variance and autocovariance of the observed versus true returns series. Of course as the sample size grows the sample statistics will approach the true statistics of the distribution (LLN). Considering the mean:

$$\tilde{\bar{r}} = \frac{1}{T}\sum_{t=1}^{T}\tilde{r}_t = \frac{1}{T}\sum_{i=1}^{j}(0+\cdots+r_{ik+1}+\cdots+r_{(i-1)k}) = \frac{1}{T}\sum_{t=1}^{T} r_t = \bar{r}$$

The mean is unaffected. This is not surprising as any missed observations are added to the next non-zero one - thus the cumulative return is preserved. Moving on to the sample variance:

$$(T-1)s_r^2 = \sum_{t=1}^{T}(r_t-\bar{r})^2 = \sum_{t=1}^{T}\left(r_t^2 - 2r_t\bar{r} + \bar{r}^2\right) = \left(\sum_{t=1}^{T} r_t^2\right) - 2\left(\sum_{t=1}^{T} r_t\right)\bar{r} + T\bar{r}^2 = \left(\sum_{t=1}^{T} r_t^2\right) - T\bar{r}^2$$

Similarly

$$(T-1)\, s_{\tilde{r}}^2 = \left(\sum_{t=1}^{T}\tilde{r}_t^2\right) - T\bar{r}^2$$

But

$$\sum_{t=1}^{T}\tilde{r}_t^2 = \sum_{i=0}^{j-1}\left(r_{ik+1}+\cdots+r_{(i-1)k}\right)^2 = \sum_{t=1}^{T} r_t^2 + \sum_{i=0}^{j-1}\sum_{m\neq n,=1}^{k} r_{ik+m}r_{ik+n}$$

So

$$(s_{\tilde{r}}^2 - s_r^2) = \frac{1}{T-1}\sum_{i=0}^{j-1}\sum_{m\neq n,=1}^{k} r_{ik+m}r_{ik+n}$$

The difference in the observed and true variance depends on the product of returns that are close together (in the same "batch"). In general this could be positive or negative. In the case where the mean is non-zero, this term will likely be higher and the variance higher. When the mean is zero, it will likely be over reported for positively serially correlated returns, and lower for negative ones. If the EMH holds (no serial correlation) and the mean is zero the sample variance should be similar.

Autocorrelation

$$(T-s)\,\gamma_r(s) = \sum_{t=s+1}^{T}(r_t-\bar{r})(r_{t-s}-\bar{r}) = \sum_{t=s+1}^{T} r_t r_{t-s} - \bar{r}\sum_{t=s+1}^{T}(r_t+r_{t-s}) + (T-s)\bar{r}^2 \approx \sum_{t=s+1}^{T} r_t r_{t-s} - (T-s)\bar{r}^2$$

because the $\sum_{t=s+1}^{T}(r_t + r_{t-s})$ term only differs from $2(T-s)\bar{r}$ by a few (s) returns at the beginning and end of the series. So

$$(T-s)\,\gamma_{\widetilde{r}}(s) \approx \sum_{t=s+1}^{T} \widetilde{r}_t \widetilde{r}_{t-s} - (T-s)\bar{r}^2$$

As $\widetilde{r}_t$ is non-zero only for multiples of k, $\widetilde{r}_t \widetilde{r}_{t-s}$ can only be non-zero when s is a multiple of k. So

$$\gamma_{\widetilde{r}}(s) \approx -\bar{r}^2 < 0, \quad \forall\, s \neq n.k \;\; n \in N$$

Thus negative auto-correlations observed auto-correlations are implied.

When $j = 1$ and $k = T$ all observed returns are zero, except the final observation which reports the total return at the end of the period. This is analogous to looking at, say, daily returns which accumulate all the intra-day data. In this case, the mean is the same (as always), the variance will differ depending on the properties of the true returns as described above and autocovariances will be negative.

The setup described in this question is of course a special case of the non-trading model where the time between reported trades (d_t in lectures) is deterministic and set to k.

Question 6

The ROLL model provides a framework to explain the presence of negative AutoCorrelation in observed trade prices at the first lag. It is very simple and relies on the observed trades occurring randomly at a fixed spread from the true "fundamental" prices.
Basic setup:

P_t	Observed Prices
P_t^*	"Fundamental" Prices
s	Fixed Spread
I_t	Trade Direction Indicator, +1 or -1 (This was Q_t in lectures).
ϵ_t	Random Walk (RW) Innovation
σ_ϵ^2	Variance of RW

I_t and ϵ_t are the Random Variables in the model and are jointly IID and mean zero. Thus:

$$E(\epsilon_t \epsilon_{t-1}) = E(\epsilon_t I_t) = E(I_t I_{t-1}) = E(\epsilon_t I_{t-1}) = E(\epsilon_t) = E(I_t) = 0$$

The ROLL model is defined as:

$$P_t = P_t^* + \frac{s}{2} I_t$$
$$\triangle P_t = \epsilon_t + \frac{s}{2} \triangle I_t$$

Considering the Covariance

$$\begin{aligned} COV(\triangle P_t^2, \triangle P_{t-1}^2) &= E(\triangle P_t^2 \triangle P_{t-1}^2) - E(\triangle P_t^2) E(\triangle P_{t-1}^2) \\ &= E(\triangle P_t^2 \triangle P_{t-1}^2) - E(\triangle P_t^2)^2 \end{aligned} \tag{1}$$

because the $\triangle P_t^2$ is stationary (ie we can change the $t-1$ to t within the $E(\cdot)$ operator). Considering only the first term:

$$E(\triangle P_t^2 \triangle P_{t-1}^2) = E\left[(\epsilon_t + \frac{s}{2}\triangle I_t)^2 \times (\epsilon_{t-1} + \frac{s}{2}\triangle I_{t-1})^2\right]$$
$$= E\left[(\epsilon_t^2 + \frac{s^2}{4}\triangle I_t^2 + \epsilon_t s \triangle I_t) \times (\epsilon_{t-1}^2 + \frac{s^2}{4}\triangle I_{t-1}^2 + \epsilon_{t-1} s \triangle I_{t-1})\right] \tag{2}$$

Note that the only terms which are not independent in the first set of brackets with the second are $\triangle I_t$ and $\triangle I_{t-1}$. As $E(A \times B) = E(A) \times E(B)$ for A & B independent and $E(\epsilon_t) = E(I_t) = 0$ means all terms with a linear ϵ or I term vanish (as the expectation factorises with a zero factor). Thus

$$E(\triangle P_t^2 \triangle P_{t-1}^2) = E\left[\epsilon_t^2 \epsilon_{t-1}^2 + \epsilon_t^2 \frac{s^2}{4}\triangle I_{t-1}^2 + \frac{s^2}{4}\triangle I_t^2 \epsilon_{t-1}^2 + \frac{s^2}{4}\triangle I_t^2 \frac{s^2}{4}\triangle I_{t-1}^2\right]$$
$$= E(\epsilon_t^2)E(\epsilon_{t-1}^2) + \frac{s^2}{4}E(\epsilon_t^2)E(\triangle I_{t-1}^2) + \frac{s^2}{4}E(\epsilon_t^2)E(\triangle I_{t-1}^2) + (\frac{s^2}{4})^2 E(\triangle I_t^2 \triangle I_{t-1}^2)$$
$$= E(\epsilon_t^2)^2 + 2 \times \frac{s^2}{4}E(\epsilon_t^2)E(\triangle I_t^2) + (\frac{s^2}{4})^2 E(\triangle I_t^2 \triangle I_{t-1}^2) \tag{3}$$

Noting that $I_t^2 = 1 \; \forall t$ and $E(I_t I_{t-1}) = 0 \Rightarrow$

$$E(\triangle I_t^2) = E(I_t^2 - 2I_t I_{t-1} + I_{t-1}^2) = 2$$

Similarly

$$E(\triangle I_t^2 \triangle I_{t-1}^2) = E\left[(I_t - I_{t-1})^2 \times (I_{t-1} - I_{t-2})^2\right]$$
$$= E\left[(I_t^2 - 2I_t I_{t-1} + I_{t-1}^2)(I_{t-1}^2 - 2I_{t-1}I_{t-2} + I_{t-2}^2)\right]$$
$$= E\left[(2 - 2I_t I_{t-1})(2 - 2I_{t-1}I_{t-2})\right]$$
$$= E\left[4 - 4I_t I_{t-1} - 4I_{t-1}I_{t-2} + 4I_t I_{t-1}^2 I_{t-2}\right]$$
$$= 4$$

Substituting back in equation 3 $\Rightarrow$

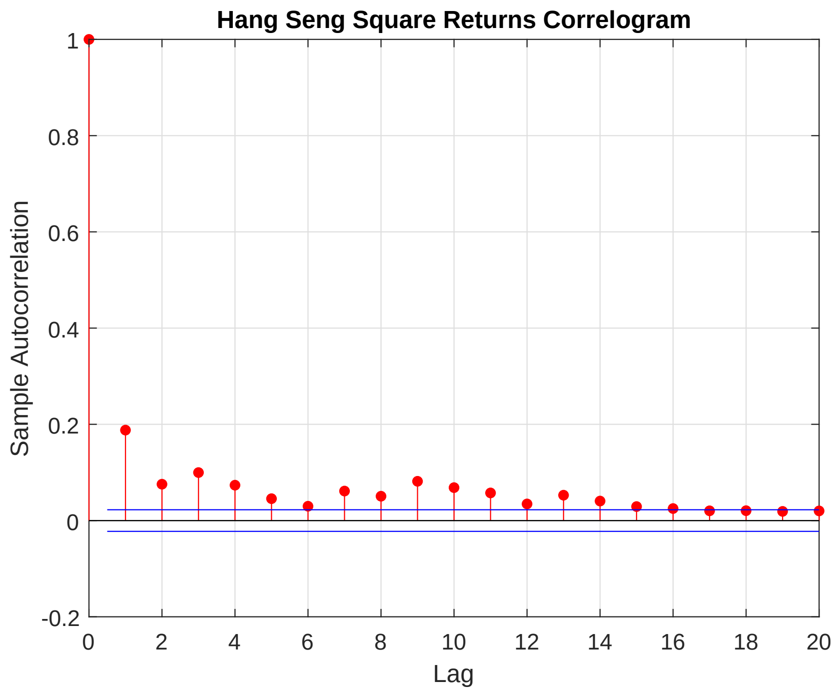
$$E(\triangle P_t^2 \triangle P_{t-1}^2) = E(\epsilon_t^2)^2 + s^2 E(\epsilon_t^2) + \frac{s^4}{4}$$

Now consider the second term in equation 1 and again using the expression for $E(\triangle I_t^2)$:

$$E(\triangle P_t^2)^2 = \left(E\left[(\epsilon_t + \frac{s}{2}\triangle I_t)^2\right]\right)^2$$
$$= \left(E\left[\epsilon_t^2 + 2 \times \frac{s}{2}\triangle I_t \epsilon_t + \frac{s^2}{4}\triangle I_t^2\right]\right)^2$$
$$= \left(E(\epsilon_t^2) + 2 \times \frac{s}{2}E(\triangle I_t)E(\epsilon_t) + \frac{s^2}{4}E(\triangle I_t^2)\right)^2$$
$$= \left(E(\epsilon_t^2) + \frac{s^2}{2}\right)^2$$
$$= E(\epsilon_t^2)^2 + s^2 E(\epsilon_t^2) + \frac{s^4}{4}$$

Which is the same as the first term and thus the Covariance is zero, as required.

This implication can be tested by examining the first term in the autocorrelation of the squared difference of price series. The implication is not reasonable as we know that volatility is autocorrelated (Daily change; $\triangle P$, is a simplest measure of "volatility" that I can think of). For the Hang Seng we have already observed that this covariance cannot be zero otherwise there would have been no adjustments in the robust errors. For completeness the ACF of the squared returns is plotted below which of course shows positive significant values.



Note that testing for the presence of this autocorrelation using robust errors makes no sense here. The result we are testing, namely that $COV(\triangle P_t^2, \triangle P_{t-1}^2) = 0$ *assumes* no serial correlation in fundamental returns. Mathematically, we use the fact that $E(\epsilon_t^2 \epsilon_{t-1}^2) = 0$ in equation 2 to derive the result.

It is also worth noting that the model application of this model to this price series is completely unreasonable as it is is based on price changes *between trades* and the Hang Seng clearly trades more than once a day.

Question 7

The model is changed to have a stochastic spread, s_t IID from the other Random Variables, with $E(s_t) = \mu$ and $VAR(s_t) = \sigma_s^2$:

$$P_t^* = P_t + \frac{s_t}{2} I_t$$
$$\triangle P_t = \epsilon_t + \frac{1}{2} \triangle (s_t I_t)$$

Because $E(\triangle P_t) = 0$

$$
\begin{aligned}
COV(\triangle P_t, \triangle P_{t-1}) &= COV\left(\epsilon_t + \frac{1}{2}\triangle(s_t I_t)\,,\, \epsilon_{t-1} + \frac{1}{2}\triangle(s_{t-1}I_{t-1})\right) \\
&= E(\frac{1}{2}\triangle(s_t I_t)\frac{1}{2}\triangle(s_{t-1}I_{t-1})) \\
&= \frac{1}{4}E((s_t I_t - s_{t-1}I_{t-1})\,(s_{t-1}I_{t-1} - s_{t-2}I_{t-2})) \\
&= -\frac{1}{4}\left[E(s_{t-1}^2 I_{t-1}^2)\right] \\
&= -\frac{1}{4}\left[E(s_{t-1}^2 \times 1)\right] \\
&= -\frac{1}{4}(\sigma^2 + \mu^2)
\end{aligned} \tag{4}
$$

Introducing the stochastic variance increases the magnitude of the non-zero autocovariance introduced by the model.

For the calculation of $COV(\triangle P_t^2, \triangle P_{t-1}^2)$ in the previous question we only used explicit terms for $E(\triangle I_t^2)$ and $E(\triangle I_t^2 \triangle I_{t-1}^2)$ along with the joint IID facts. Replacing s with 1 and I_t by the random variable $\widetilde{I}_t = s_t I_t$ in question 6 does not change the IID properties but now:

$$
\begin{aligned}
E(\triangle\widetilde{I}_t^2) &= E\left[(s_t I_t - s_{t-1}I_{t-1})\right]^2 \\
&= E\left[s_t^2 I_t^2 + s_{t-1}^2 I_{t-1}^2 - 2 s_t s_{t-1} I_t I_{t-1}\right] \\
&= 2 \times E(s_t^2) \\
&\left[= 2 \times (\sigma^2 + \mu^2)\right]
\end{aligned} \tag{5}
$$

And

$$
\begin{aligned}
E(\triangle\widetilde{I}_t^2 \triangle\widetilde{I}_{t-1}^2) &= E\left[(s_t I_t - s_{t-1}I_{t-1})^2 \times (s_{t-1}I_{t-1} - s_{t-2}I_{t-2})^2\right] \\
&= E\left[(s_t^2 I_t^2 - 2 s_t I_t s_{t-1} I_{t-1} + s_{t-1}^2 I_{t-1}^2)(s_{t-1}^2 I_{t-1}^2 - 2 s_{t-1} I_{t-1} s_{t-2} I_{t-2} + s_{t-2}^2 I_{t-2}^2)\right] \\
&= E\left[(s_t^2 + s_{t-1}^2 - 2 s_t I_t s_{t-1} I_{t-1})(s_{t-1}^2 + s_{t-2}^2 - 2 s_{t-1} I_{t-1} s_{t-2} I_{t-2})\right] \\
&= E\left[s_t^2 s_{t-1}^2 + s_t^2 s_{t-2}^2 + s_{t-1}^4 + s_t^2 s_{t-2}^2 +\right] \\
&= 3 \times E(s_t^2)^2 + E(s_t^4)
\end{aligned} \tag{6}
$$

Substituting the expressions in equations 5 and 6 above into equation 3 in question 6 yields

$$
\begin{aligned}
E(\triangle P_t^2 \triangle P_{t-1}^2) &= E(\epsilon_t^2)^2 + E(\epsilon_t^2)E(s_t^2) + (\frac{1}{4})^2\left(3 \times E(s_t^2)^2 + E(s_t^4)\right) \\
&= E(\epsilon_t^2)^2 + E(\epsilon_t^2)E(s_t^2) + \frac{3}{16}E(s_t^2)^2 + \frac{1}{16}E(s_t^4)
\end{aligned}
$$

And substituting $E(\triangle\widetilde{I}_t^2)$ into the second term of equation 3 yields

$$
\begin{aligned}
E(\triangle P_t^2)^2 &= \left(E(\epsilon_t^2) + \frac{1}{4}E(\triangle\widetilde{I}_t^2)\right)^2 \\
&= \left(E(\epsilon_t^2) + \frac{E(s^2)}{2}\right)^2 \\
&= E(\epsilon_t^2)^2 + E(\epsilon_t^2)E(s_t^2) + \frac{E(s^2)^2}{4}
\end{aligned}
$$

The difference is thus:

$$COV(\triangle P_t^2, \triangle P_{t-1}^2) = \frac{1}{16}\left(E(s_t^4) - E(s_t^2)^2\right) > 0$$

Interestingly including a stochastic IID spread introduces serially correlated variance in the observed square returns, even though there is no serial correlation in the spread. Note that this autocorrelation is in fact related to the fourth moment (kurtosis) of the distribution of s (being proportional to the variance of s^2).

When s_t is deterministic, return to equation 4 above:

$$COV(\triangle P_t, \triangle P_{t-1}) = -\frac{1}{4}\left[E(s_{t-1}^2 I_{t-1}^2)\right] = -\frac{1}{4}s_{t-1}^2$$

as required (somewhat easier than the previous part!).

The magnitude of the autocovariance at lag 1 in the model increases with the spread, thus around times of wider spread (ie the open and close) the observed *trade* series should be more predictable. However, even though the model may make this prediction it does not mean one can profit from it - you require the possession of the monopoly power of the market maker! Everyone else can only trade by crossing the spread which is in the exact opposite direction as to the direction you want to trade! This illustrates a more general point about this model in that the autocorrelation is simply a facet of the market maker being able to execute trades at a spread either side of the "fundamental price" to her advantage.

Question 8

Book work...but of course do not forget to include the 7 steps in an event study...