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Semiparametric identification of the bid–ask spread in extended Roll models

Xiaohong Chen [a], Oliver Linton [b], Yanping Yi [c,*]

[a] Cowles Foundation for Research in Economics, Yale University, PO Box 208281, New Haven, CT 06520-8281, USA
[b] Department of Economics, University of Cambridge, Austin Robinson Building, Sidgwick Avenue, Cambridge, CB3 9DD, UK
[c] School of Economics, Shanghai University of Finance and Economics, 777 Guoding Road, Shanghai, 200433, China

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ABSTRACT

This paper provides new identification results for the bid–ask spread and the nonparametric distribution of the latent fundamental price increments (ε_t) from the observed transaction prices alone. The results are established via the characteristic function approach, and hence allow for discrete or continuous ε_t and the observed price increments do not need to have any finite moments. Constructive identification (and overidentification) results are established first in the basic Roll (1984) model, and then in various extended Roll models, including general unbalanced order flow, serially dependent latent trade direction indicators, adverse selection, random spread and a multivariate Roll model.

1. Introduction

The (quoted) bid–ask spread of a financial asset is the difference between the best quoted prices for an immediate purchase and an immediate sale of that asset. The spread represents a potential profit for the market maker handling the transaction, and is a major part of the transaction cost facing investors, especially since the elimination of commissions and the reduction in exchange fees that has happened in the last twenty years; see for example Jones (2002) and Angel et al. (2011). Measuring the bid–ask spread in practice can be quite time consuming (since it requires reconstruction of the limit order book) and may be subject to a number of potential accuracy issues due to the quoting strategies of High Frequency Traders, for example.

The seminal paper Roll (1984) provides a simple market microstructure model that allows one to estimate the bid–ask spread from observed transaction prices alone, without information on the underlying bid–ask price quotes and the order flow (i.e., whether a trade was buyer- or seller-induced). This is particularly useful for long historical data sets, which are often limited in their scope. For instance, Hasbrouck (2009) notes that "investigations into the role of liquidity and transaction costs in asset pricing must generally confront the fact that while many asset pricing tests make use of US equity returns from 1926 onward, the high-frequency data used to estimate trading costs are usually not available prior to 1983. Accordingly, most studies either limit the sample to the post-1983 period of common coverage or use the longer historical sample with liquidity proxies estimated from daily data". Another area where the available data is limited is open-outcry markets (like the CME), in which bid and ask quotes by traders expire (if not filled) without recording (see, e.g., Hasbrouck (2004) for more details).

In the famous Roll (1984) model, an observed (log) asset price p_t evolves according to

$$p_t = p_t^* + I_t \frac{s_0}{2}, \quad p_t^* = p_{t-1}^* + \varepsilon_t. \tag{1}$$

$$\Delta p_t := p_t - p_{t-1} = \varepsilon_t + (I_t - I_{t-1})\frac{s_0}{2}, \tag{2}$$

where p_t^* is the underlying fundamental (log) price with innovations ε_t, and the trade direction indicators $\{I_t\}$ are i.i.d. and take the values ± 1 with probability $q_0 := \Pr(I_t = 1) = 1/2$. $I_t = 1$ indicates that the transaction is a purchase, and $I_t = -1$ denotes a sale. The price p_t is observed, whereas *all other variables in* Eq. (1) *are unobserved*. The parameter of interest is the effective bid–ask spread s_0.[1] Roll (1984) assumes that $\{\varepsilon_t\}$ is serially uncorrelated and uncorrelated with the trade direction indicators $\{I_t\}$, and that

[1] The bid–ask spread in Eq. (1) is called effective bid–ask spread because it is based on the effective (average) price p_t that is paid to fill an order, and not necessarily on the quoted bid or ask price, since it might be the case that the order cannot be filled at the latter price (e.g., due to insufficient depth of the market).

* Corresponding author.
E-mail addresses: xiaohong.chen@yale.edu (X. Chen), obl20@cam.ac.uk (O. Linton), yi.yanping@mail.shufe.edu.cn (Y. Yi).

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the one period returns (i.e., the price increments) $\{\Delta p_t\}$ have finite second moments. Under these assumptions, s_0 is identified in a closed form as

$$s_0 = 2\sqrt{-\operatorname{Cov}(\Delta p_t, \Delta p_{t-1})}. \tag{3}$$

Roll (1984) proposes to estimate s_0 from (3) by replacing the theoretical covariance by its empirical counterpart, i.e.,

$$\widehat{s}_{Roll} := 2\sqrt{-\widehat{\operatorname{Cov}}(\Delta p_t, \Delta p_{t-1})}. \tag{4}$$

In practice, this estimator is not satisfactory, since the empirical first-order autocovariance of price changes is often positive, in which case (4) is not well-defined. Another problem is that the nonparametric distribution of the latent true one period returns (i.e., the latent fundamental price increment), $\Delta p_t^* = \varepsilon_t$, is not identifiable in the original Roll model.

In a well-known alternative, Hasbrouck (2004) proposes to strengthen Roll's modelling assumptions by assuming that $\{\varepsilon_t\}$ is i.i.d. with a known parametric distribution, and is independent of $\{I_t\}$.[2] He then uses a Bayesian Gibbs sampling methodology to estimate the spread parameter subject to a non-negativity constraint. Specifically, Hasbrouck (2004) assumes that $\varepsilon_t \sim$ i.i.d.$N(0, \sigma_\varepsilon^2)$, where the parameter σ_ε is estimated jointly with the spread s_0. Unfortunately the spread estimator of Hasbrouck (2004) performs poorly or is not well defined when ε_t is discrete or continuous but fat-tailed and/or asymmetric. Basically the spread estimator of Hasbrouck (2004) is very sensitive to departures from the assumption that $\varepsilon_t \sim$ i.i.d.$N(0, \sigma_\varepsilon^2)$. Moreover, it is difficult to justify a specific parametric distribution such as Gaussian for the latent ε_t.

The more recent empirical finance literature emphasizes several additional issues with the Roll model: (a) It assumes balanced market order flow, i.e., $q_0 = 1/2$, which may be accurate on average, but may be inaccurate for certain episodes of trading. (b) It assumes no serial correlation in trade direction indicators, i.e., I_t is uncorrelated with I_{t-j} for any $j \geq 1$. (c) Market orders are assumed not to bring any news into the fundamental prices (i.e., no adverse selection), so that I_t is uncorrelated with Δp_{t+j}^* for $j \geq 0$. (d) Spreads are constant within the sample period. Admitting any one of these effects in the model will lead to the undesired consequence that the spread estimators of Roll (1984) and Hasbrouck (2004) become *inconsistent* (i.e., biased even as sample size goes to infinity). Furthermore, without additional model assumptions, or additional observed information (such as trade volume data in addition to $\{p_t\}$), it may not be possible to identify the spread jointly with parameters describing order flow imbalance or adverse selection, for example. See, e.g., Bleaney and Li (2015) for a very recent discussion of all the above and additional problems with the original Roll model.

In this paper we propose new methods for identifying the bid–ask spread s_0 and the unknown distribution of $\{\varepsilon_t\}$ jointly from the observed time series transaction prices alone. The observed prices $\{p_t\}$ could be daily or weekly closing prices, or high-frequency intra-day prices. Our methods are based on the characteristic function approach, and hence do not require the existence of any finite moments of $\{\Delta p_t\}$, and allow the latent $\{\varepsilon_t\}$ to be discrete or continuous, symmetric or asymmetric. Under the assumption of strict stationarity of the latent process $\{\varepsilon_t, I_t\}_{t=1}^\infty$, our identification results do not require the full independence between $\{\varepsilon_t\}$ and $\{I_t\}$, and mainly impose some restrictions on the dependence structure of ε_t, ε_{t-1}, I_t, I_{t-1} and I_{t-2}. Constructive identification results for s_0 and the characteristic function (φ_ε) of ε_t or/and parameters in various extended Roll models are established based on the joint characteristic function of consecutive one period returns

$$\varphi_{\Delta p,2}(u, u') := \mathbb{E}\left[\exp\left(iu\Delta p_t + iu'\Delta p_{t-1}\right)\right] \quad \text{for any } (u, u') \in \mathbb{R}^2, \tag{5}$$

which is nonparametrically identified from the observed price increment time series $\{\Delta p_t\}$.

We first provide a closed-form solution of $(s_0, \varphi_\varepsilon)$ in the basic Roll (1984) model under a mild sub-independence assumption, which is only slightly stronger than the uncorrelatedness condition in Roll (1984) but is much weaker than the full independence between $\{\varepsilon_t\}$ and $\{I_t\}$ assumption in Hasbrouck (2004). In addition, we do not impose finite second moment of Δp_t as in Roll (1984) and Gaussian error of ε_t as in Hasbrouck (2004). We then propose solutions to the four problems (a)–(d) with the Roll model listed above. We show how to identify $(s_0, \varphi_\varepsilon)$ and other parameters associated with unbalanced order flow and/or general asymmetric supported $\{I_t\}$, or those for serially correlated $\{I_t\}$, or those capturing adverse selection effects, or the random spread. We also extend the basic Roll model to the multivariate case and derive the identification results. Again, all these are accomplished without requiring additional data.

In principle, both the basic Roll (1984) model and the various extended Roll models could fit into the vast measurement error literature (see, e.g., Li and Vuong, 1998; Carroll et al., 2006; Hu, 2008; Hu and Schennach, 2008; Chen et al., 2011; Evdokimov and White, 2012; Bonhomme et al., 2016; Hu, forthcoming, and the references therein). However, to the best of our knowledge, our identification results are not direct consequences of any existing published results. This is because the Roll model and its various extensions contain some special structures, and our identification results utilize these special features and are constructive under conditions reasonable for financial applications.

Our constructive identification results for $(s_0, \varphi_\varepsilon)$ or/and parameters in extended Roll models are derived under conditions much weaker than those in the existing literature and more realistic for financial applications when $\{p_t\}$ is the only information available. All our identification results are essentially based on solving the unknown model parameters by matching the nonparametrically identified characteristic function $\varphi_{\Delta p,2}(u, u')$ to its model-implied semiparametric counterpart. This approach actually leads to Hansen (1982) style overidentification.[3] Therefore, one could easily compute consistent estimators of s_0, the distribution of ε_t or/and other model parameters via minimum distance procedures based on empirical characteristic functions. And the overidentification restrictions allow for model specification tests. As a natural follow-up to this identification paper, Chen et al. (2017) studies in detail the estimation and testing aspects of these models and presents an interesting empirical application. In particular, based on our constructive identification results, Chen et al. (2017) provides simple sample analog estimation of the spread s_0, the characteristic function of ε_t or/and other parameters in various extended Roll models (such as order flow imbalance, adverse selections). In the simulation studies, their sample analog spread estimator does not suffer the pitfalls of the spread estimators of Roll (1984) and Hasbrouck (2004).

The rest of the paper is organized as follows: Section 2 presents the basic Roll model and identification of both the spread s_0 and the characteristic function of ε_t in closed form, allowing for $\{\Delta p_t\}$ to have infinite first absolute moments. Section 3 considers extensions to models that allow for unbalanced order flow and more general asymmetric supported $\{I_t\}$. Section 4 studies identification

2 Hasbrouck (2004) presents an extension that relaxes the independence between $\{\varepsilon_t\}$ and $\{I_t\}$ assumption but uses additional trade volume data.

3 See Chen and Santos (2015) for a notion of overidentification in semiparametric and nonparametric models.

in models with serially dependent $\{I_t\}$. Section 5 addresses the effects of a market order on the latent fundamental price. Section 6 considers identification in models with possibly random spread. Section 7 extends the basic Roll model to a multivariate case. Section 8 concludes. The Appendix contains proofs that are not presented in the main text.

2. Identification in basic roll models

This section presents identification (and overidentification) results in a basic Roll (1984) type model satisfying the following Assumption.

Assumption 1 (*Basic Roll*)**.** (i) Data $\{p_t\}_{t=1}^T$ is generated from Eq. (1) with $s_0 > 0$, where $\{\varepsilon_t, I_t\}_{t=1}^\infty$ is a strictly stationary process; (ii) $\{I_t\}$ has marginal distribution that takes the values ± 1 with equal probability.

Throughout the paper we do not impose any restriction on the distribution of ε_t. It could be discrete and could have no finite moments, and its characteristic function (c.f.), $\varphi_\varepsilon(u) := \mathbb{E}[\exp(iu\varepsilon_t)]$, could have many zeros.

2.1. Diagonal identification

We first introduce the notion of sub-independence, which is weaker than independence.

Definition 1 (*Sub-independence*)**.** Real-valued random variables X and Y are sub-independent if for all $t \in \mathbb{R}$

$$\mathbb{E}[\exp(it(X+Y))] = \mathbb{E}[\exp(itX)]\mathbb{E}[\exp(itY)], \quad \text{where } i = \sqrt{-1}.$$

Sub-independence amounts to a restriction only on the diagonal of the joint characteristic function. It is a stronger restriction than uncorrelatedness, but strictly weaker than independence.[4] See Ebrahimi et al. (2010) and Hamedani (2013) and the references therein for detailed discussion of the notion of sub-independence. Schennach (2013) argues that it is similar to a conditional moment restriction. We make the following assumption.

Assumption 2 (*Sub-independence*)**.** (i) ε_t is sub-independent of $(I_t - I_{t-1})\frac{s_0}{2}$; I_t is sub-independent of $-I_{t-1}$; (ii) $\varepsilon_t + \varepsilon_{t-1}$ is sub-independent of $(I_t - I_{t-2})\frac{s_0}{2}$; I_t is sub-independent of $-I_{t-2}$; and ε_t is sub-independent of ε_{t-1}.

This assumption is enough for identification for the basic Roll model. But it might be simpler to replace the conditions that ε_t is sub-independent of $(I_t - I_{t-1})\frac{s_0}{2}$ and $\varepsilon_t + \varepsilon_{t-1}$ is sub-independent of $(I_t - I_{t-2})\frac{s_0}{2}$ by their stronger versions that ε_t is independent of $(I_t - I_{t-1})$ and $\varepsilon_t + \varepsilon_{t-1}$ is independent of $(I_t - I_{t-2})$ respectively.

Let $\varphi_{\Delta p,1}(u) := \mathbb{E}[\exp(iu\Delta p_t)]$ be the marginal c.f. of one period returns Δp_t, and $\varphi_{\Delta^2 p}(u) := \mathbb{E}\left[\exp\left(iu\Delta^2 p_t\right)\right]$ be the marginal c.f. of two period returns $\Delta^2 p_t := p_t - p_{t-2}$. By definition, $\varphi_{\Delta p,1}(u) \equiv \varphi_{\Delta p,2}(u, 0)$ and $\varphi_{\Delta^2 p}(u) \equiv \varphi_{\Delta p,2}(u, u)$, and are nonparametrically identified from data.

Let $\varphi_I(u) := \mathbb{E}[\exp(iuI_t)]$ be the c.f. of I_t. Under Assumptions 1(i) and 2(i), the c.f. of one period returns, $\Delta p_t = \varepsilon_t + (I_t - I_{t-1})\frac{s_0}{2}$, satisfies

$$\varphi_{\Delta p,1}(u) = \varphi_\varepsilon(u)\varphi_I\left(u\frac{s_0}{2}\right)\varphi_I\left(-u\frac{s_0}{2}\right) \quad \text{for all } u \in \mathbb{R}. \tag{6}$$

Under Assumptions 1(i) and 2(ii), the c.f. of two period returns, $\Delta^2 p_t = \varepsilon_t + \varepsilon_{t-1} + (I_t - I_{t-2})\frac{s_0}{2}$, satisfies

$$\varphi_{\Delta^2 p}(u) = [\varphi_\varepsilon(u)]^2\varphi_I\left(u\frac{s_0}{2}\right)\varphi_I\left(-u\frac{s_0}{2}\right) \quad \text{for all } u \in \mathbb{R}. \tag{7}$$

Denote

$$\overline{\mathcal{V}} := \left\{u \in \mathbb{R} : \varphi_{\Delta p,1}(u) \neq 0\right\}. \tag{8}$$

Since $\varphi_{\Delta p,1}(\cdot)$ is uniformly continuous on $\mathbb{R}$ (see, e.g., page 3 of Lukacs (1972)) and $\varphi_{\Delta p,1}(0) = 1$, the set $\overline{\mathcal{V}}$ contains an open interval of 0. This fact will be used repeatedly in the paper.

Eqs. (6) and (7) immediately imply that the c.f. $\varphi_\varepsilon(\cdot)$ is identified.

Theorem 1. *Let Assumptions 1(i) and 2 hold. Then the c.f. $\varphi_\varepsilon(\cdot)$ is identified as*

$$\varphi_\varepsilon(u) = \frac{\varphi_{\Delta^2 p}(u)}{\varphi_{\Delta p,1}(u)}, \quad \forall u \in \overline{\mathcal{V}}. \tag{9}$$

This theorem states that $\varphi_\varepsilon(\cdot)$ is identified on $\overline{\mathcal{V}}$ under very mild conditions, regardless whether s_0 and $\varphi_I(\cdot)$ are known or not.

We next consider identification of s_0. Eqs. (6) and (7) and the definition of $\overline{\mathcal{V}}$ imply that: for all $u \in \overline{\mathcal{V}}$ we have $\varphi_\varepsilon(u) \neq 0$, $\varphi_I\left(u\frac{s_0}{2}\right)\varphi_I\left(-u\frac{s_0}{2}\right) \neq 0$ and $\varphi_{\Delta^2 p}(u) \neq 0$. Denote

$$h(u) := \frac{\varphi^2_{\Delta p,1}(u)}{\varphi_{\Delta^2 p}(u)} \quad \text{for any } u \in \overline{\mathcal{V}}, \tag{10}$$

which is continuous on $\overline{\mathcal{V}}$ with $h(0) = 1$, and nonparametrically identified from the data $\{\Delta p_t\}$. Moreover, Eqs. (6) and (7) imply that

$$h(u) = \varphi_I\left(u\frac{s_0}{2}\right)\varphi_I\left(-u\frac{s_0}{2}\right) \quad \text{for all } u \in \overline{\mathcal{V}}. \tag{11}$$

Since I_t is a discrete random variable, the c.f. $\varphi_I(\cdot)$ is analytic in $u \in \mathbb{R}$. Eq. (11) implies that $h(u)$ is analytic in $\overline{\mathcal{V}}$, and hence $\frac{d^2h(u)}{du^2}$ is well-defined in $u \in \overline{\mathcal{V}}$ and satisfies[5]

$$\frac{d^2h(0)}{du^2} = -\frac{s_0^2}{2}Var(I_t). \tag{12}$$

Eq. (12) would lead to the global identification of $s_0 > 0$ as soon as $Var(I_t)$ is known. This is similar to the closed form solution (3) for s_0 originally proposed in Roll (1984).

Under additional Assumption 1(ii) (i.e., balanced order flow), we have $\varphi_I(u) = \cos(u)$ for all $u \in \mathbb{R}$ and $Var(I_t) = 1$, and hence Eq. (11) becomes

$$h(u) = \left[\cos\left(u\frac{s_0}{2}\right)\right]^2 \quad \text{for all } u \in \overline{\mathcal{V}}. \tag{13}$$

This immediately identifies the unknown true spread $s_0 > 0$, as stated in the following theorem.

Theorem 2. *Let Assumptions 1 and 2 hold. Then: for some non-zero $\widetilde{u} \in \overline{\mathcal{V}}$, the true spread s_0 is locally identified as $\left\{\left|\frac{2}{\widetilde{u}}\left[\arccos\left(\sqrt{h(\widetilde{u})}\right) \pm \pi j\right]\right|, j = 0, 1, 2, \ldots\right\}$.*

(1) If it is known that $s_0 \in \mathcal{S} := [0, \bar{s}]$ for some finite $\bar{s}$, then s_0 is globally identified in $\mathcal{S}$ as

$$s_0 = \frac{2}{\widetilde{u}}\arccos\left(\sqrt{h(\widetilde{u})}\right) \quad \textit{for some } \widetilde{u} \in (0, \pi/\bar{s}) \cap \overline{\mathcal{V}}.$$

(2) s_0 is globally identified in $\mathbb{R}_+$ as $s_0 = \sqrt{-2\frac{d^2h(0)}{du^2}}$.

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4 Recall that real-valued random variables X and Y are independent if $\mathbb{E}[\exp(i(tX + sY))] = \mathbb{E}[\exp(itX)]\mathbb{E}[\exp(isY)]$ for all $t, s \in \mathbb{R}$.

5 By definition (10) of $h(\cdot)$ and without invoking Eq. (11), one sufficient condition for a twice-differentiable $h(\cdot)$ is to assume that $\varphi_{\Delta p,1}(\cdot)$ and $\varphi_{\Delta^2 p}(\cdot)$ are twice differentiable. However, the twice-differentiability of these characteristic functions requires that $E[|\Delta p_t|^2] < \infty$ (see, e.g., Theorem 1.2. of Lukacs (1972)), which would exclude some distributions such as Cauchy.

Theorem 2 provides two closed form identification results for s_0. One could estimate s_0 by sample analog principle based on either Theorem 2 part (1) or part (2). However, the sample analog estimation of s_0 based on Theorem 2 part (2) will not perform well in practice since it involves nonparametric estimation of second derivative of $h(\cdot)$. In financial applications we expect s_0 to be a small positive value. Therefore, the restriction $s_0 \in \mathcal{S}$ is very natural and the sample analog estimation of s_0 based on Theorem 2 part (1) is easy to compute as well. In the rest of the paper we maintain the assumption $s_0 \in \mathcal{S}$ and present identification results similar to Theorem 2 part (1).

We next present an alternative identification result for $(s_0, \varphi_\varepsilon)$ under slightly different conditions, which are weaker in some respects but stronger in other respects. Under Assumptions 1(i)(ii) and 2(i), Eq. (6) becomes

$$\varphi_{\Delta p,1}(u) = \varphi_\varepsilon(u)\left[\cos\left(u\frac{s_0}{2}\right)\right]^2 \text{ for all } u \in \mathbb{R}. \tag{14}$$

This relation immediately leads to the following result.

Proposition 1. *Let Assumptions 1 and 2(i) hold. Suppose that $|\varphi_\varepsilon(u)| > 0$ for all $u \in \mathbb{R}$. Denote $u_0 := \inf\{u > 0 : \varphi_{\Delta p,1}(u) = 0\}$. Then:*

(1) s_0 can be identified as the unique element in $\mathcal{S}$ satisfying $s_0 = \pi/u_0$.

(2) φ_ε can be identified as $\varphi_\varepsilon(u) = \varphi_{\Delta p,1}(u)\left[\cos\left(\frac{\pi u}{2u_0}\right)\right]^{-2}$ on $\overline{\mathcal{V}}$.

Proposition 1 does not impose Assumption 2(ii) and hence allows quite general forms of temporal dependence in $\{\varepsilon_t\}$. It does not restrict the joint distribution of $(\varepsilon_t, \varepsilon_{t-1})$ at all. However, it requires stronger restrictions on the c.f. $\varphi_\varepsilon(\cdot)$ of the latent ε_t. This condition would be satisfied by Normal or Cauchy errors, but would not be satisfied by the uniform distribution, for example, nor would it be satisfied by any discrete distribution. In high frequency financial applications, $\Delta p_t^* = \varepsilon_t$ often contains discrete components. It is possible to weaken the condition that $|\varphi_\varepsilon(u)| > 0$ for all $u \in \mathbb{R}$ to the requirement that this holds over a large compact set, but then it would need some side information to resolve the location of zeros of $\varphi_\varepsilon(\cdot)$ from zeros implied by the parametric part in Eq. (14).

2.2. Off-diagonal information

Theorem 2 part (1) already obtains overidentification of the spread parameter s_0 by considering a set of values of $u \in (0, \pi/\bar{s}] \cap \overline{\mathcal{V}}$. We next show how to use additional restrictions from the joint c.f. of consecutive one period returns $\varphi_{\Delta p,2}$ (defined in (5)).

In the rest of the paper we make use of the following definition repeatedly. Let

$$H(u, u') := \frac{\varphi_{\Delta p,2}(u, u')}{\varphi_{\Delta p,1}(u)\varphi_{\Delta p,1}(u')} \quad \text{for any } (u, u') \in \overline{\mathcal{V}}^2, \tag{15}$$

which is continuous on $\overline{\mathcal{V}}^2$ with $H(0, 0) = 1$, and is nonparametrically identified from the data $\{\Delta p_t\}$.

Note that $\varphi_{\Delta^2 p}(u) \equiv \varphi_{\Delta p,2}(u, u)$, the marginal c.f. of two period returns is found on the diagonal of the joint c.f. $\varphi_{\Delta p,2}$. We now seek to exploit restrictions on the off-diagonal elements where $u \neq u'$. Let $\Delta I_t := I_t - I_{t-1}$.

Assumption 3. (i) $(\varepsilon_t, \varepsilon_{t-1})$ is independent of $(\Delta I_t, \Delta I_{t-1})$; (ii) ε_t is independent of ε_{t-1}; and (iii) I_t, I_{t-1} and I_{t-2} are independent.

Note that Assumption 3 is stronger than Assumption 2, but is weaker than the full independence condition.

Under Assumptions 1 and 3, for all $(u, u') \in \mathbb{R}^2$ we have:

$$\varphi_{\Delta p,2}(u, u') = \varphi_\varepsilon(u)\varphi_\varepsilon(u')\cos\left(u\frac{s_0}{2}\right)\cos\left((u'-u)\frac{s_0}{2}\right)\cos\left(u'\frac{s_0}{2}\right). \tag{16}$$

Denote

$$\overline{\mathcal{U}} := \left\{(u, u') \in \overline{\mathcal{V}} \times \overline{\mathcal{V}} : \min_{s \in \mathcal{S}}\left|\cos\left(u\frac{s}{2}\right)\cos\left(u'\frac{s}{2}\right)\right| > 0\right\}. \tag{17}$$

Let

$$R(u, u'; s) := \frac{\cos\left((u-u')\frac{s}{2}\right)}{\cos\left(u\frac{s}{2}\right)\cos\left(u'\frac{s}{2}\right)}, \tag{18}$$

which is well defined on $\overline{\mathcal{U}} \times \mathcal{S}$. Eq. (16) implies that

$$H(u, u') = R(u, u'; s_0) \text{ for all } (u, u') \in \overline{\mathcal{V}}^2, \tag{19}$$

and hence $H(u, u')$ is analytic and real-valued for all $(u, u') \in \overline{\mathcal{V}}^2$. Eq. (19) is free of the nuisance function $\varphi_\varepsilon(\cdot)$ and only depends on the parameter of interest s_0, which is the key insight of our alternative overidentification methods.

Due to the continuity of the c.f. $\varphi_{\Delta p,2}(u, u')$ in $\mathbb{R}^2$ and $\varphi_{\Delta p,2}(0, 0) = 1$, the set $\overline{\mathcal{V}}^2$ (and hence $\overline{\mathcal{U}}$) contains an open ball of $(0, 0)$, and hence Eq. (19) contains infinitely many overidentifying restrictions for s_0. Let $\mathcal{U} \subseteq \overline{\mathcal{U}}$ and $|\mathcal{U}|$ denote the number of points in $\mathcal{U}$, which can be chosen such that $|\mathcal{U}| \geq 1$. We introduce a simple population minimum distance criterion function on $\mathcal{S}$:[6]

$$Q(s, \mathcal{U}) := \sum_{(u,u') \in \mathcal{U}} |H(u, u') - R(u, u'; s)|^2 \geq 0. \tag{20}$$

Here, $|\cdot|$ denotes the modulus of a complex number i.e., $|a + bi|^2 = a^2 + b^2$. Since Eq. (19) holds for all $(u, u') \in \overline{\mathcal{V}}^2$ and $\mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \overline{\mathcal{V}}^2$, $Q(s, \mathcal{U})$ is minimized at $s = s_0$, i.e., $Q(s_0, \mathcal{U}) = 0$.

Assumption 4. (i) $s_0 \in \mathcal{S}$; (ii) either (a) $\mathcal{U} = \overline{\mathcal{U}}$; or (b) $\mathcal{U} \subset \overline{\mathcal{U}}$, and $\exists(\tilde{u}, \tilde{u}) \in \mathcal{U}$ such that $\tilde{u} \in (0, \pi/\bar{s})$.

We present an alternative identification for s_0 below.

Theorem 3. *Let Assumptions 1, 3 and 4 hold. Then: s_0 is identified as the unique solution to $\min_{s \in \mathcal{S}} Q(s, \mathcal{U})$, and satisfies the identifiable uniqueness on $\mathcal{S}$.*[7]

The proof of Theorem 3 is relegated to the Appendix. As shown in Theorem 2 part (1), for the identification of s_0 it suffices to choose a grid $\mathcal{U}$ satisfying Assumption 4(ii)(b) with $|\mathcal{U}| = 1$. But a grid $\mathcal{U}$ with larger $|\mathcal{U}| > 1$ is better for more accurate estimation of s_0. Theorem 3 suggests a natural minimum distance estimation procedure for s_0.

3. Models with general unbalanced order flow

This section presents identification results for two extended Roll models that relax Assumption 1(ii) (i.e., balanced order flow) imposed in the basic Roll model.

We maintain Assumptions 1(i) and 3 in this section, which implies that for all $(u, u') \in \mathbb{R}^2$,

$$\varphi_{\Delta p,2}(u, u') = \varphi_\varepsilon(u)\varphi_\varepsilon(u')\varphi_I\left(u\frac{s_0}{2}\right)\varphi_I\left((u'-u)\frac{s_0}{2}\right)\varphi_I\left(-u'\frac{s_0}{2}\right). \tag{21}$$

6 If $|\mathcal{U}| = \infty$, there is a slight abuse of notations in definition (20). Summations should be replaced by integrals with respect to some (positive) sigma-finite measure on $\mathcal{U}$.

7 That is, for all sequences $\{a_k\} \subset \mathcal{S}$ with $Q(a_k, \mathcal{U})$ going to 0, we have $|a_k - s_0|$ goes to zero.

Thus Theorem 1 remains valid, and the c.f. $\varphi_\varepsilon(\cdot)$ is still identified as (9) on $\overline{\mathcal{V}}$.

Eq. (21) also implies the following identification relation for $(s_0, \varphi_I(\cdot))$:

$$H(u,u') = \frac{\varphi_I\left((u'-u)\frac{s_0}{2}\right)}{\varphi_I\left(-u\frac{s_0}{2}\right)\varphi_I\left(u'\frac{s_0}{2}\right)} \quad \text{for all } (u,u') \in \overline{\mathcal{V}}^2 . \tag{22}$$

Since I_t is a discrete random variable, $\varphi_I(\cdot)$ is analytic, and hence $H(u,u')$ is analytic in $(u,u') \in \overline{\mathcal{V}}^2$.

Note that Eq. (12) remains valid without imposing Assumption 1(ii), and would lead to global identification of s_0 as soon as $Var(I_t)$ is identified. However, we need the off-diagonal information contained in Eq. (22) for the identification of the parameters of the probability distribution of I_t in general unbalanced order flow situations.

3.1. Unbalanced order flow

Assumption 5. $\{I_t\}$ takes values ± 1 with unknown probability $q_0 := \Pr(I_t = 1) \in (0, 1)$.

This relaxation of Assumption 1(ii) allows for unbalanced order flow (i.e., $q_0 \neq 1/2$). Assumption 5 implies that the c.f. $\varphi_I(\cdot)$ of I_t takes the form

$$\varphi_I(u) = \cos(u) + (2q_0 - 1) \times i \sin(u) \quad \text{for all } u \in \mathbb{R} , \tag{23}$$

and $Var(I_t) = 1 - (2q_0 - 1)^2$.

Eqs. (21) (or (22)) and (23) imply the following identification relation for (s_0, q_0):

$$H(u,u') = R(u,u'; s_0, q_0) \text{ for all } (u,u') \in \overline{\mathcal{V}}^2, \tag{24}$$

where $R(u,u'; s, q)$ (given in (53) in the Appendix) is a parametric function defined on $\overline{\mathcal{U}} \times \mathcal{S} \times (0, 1)$. When $q_0 = 1/2$ we have $R(u,u'; s, 1/2) = R(u,u'; s)$ defined in (18), and Eq. (24) becomes the identification relation (19) for s_0 in Section 2.

Assumption 6. (i) $s_0 \in \mathcal{S}$; (ii) either (a) $\mathcal{U} = \overline{\mathcal{U}}$; or (b) $\mathcal{U} \subset \overline{\mathcal{U}}$, and $\exists(\tilde{u}, \tilde{u}), (\tilde{u}, -\tilde{u}) \in \mathcal{U}$ such that $\tilde{u} \in (0, \pi/\bar{s})$.

Theorem 4. *Let Assumptions 1(i), 3 and 5 hold. Then:*

(1) q_0 is identified by Eqs. (55) and (56) (in the Appendix) with a small positive $\tilde{u} \in \overline{\mathcal{V}}$ and $s_0 > 0$ is identified via Eq. (12). If $s_0 \in \mathcal{S}$ then s_0 is also identified by Eq. (54) (in the Appendix) with a $\tilde{u} \in (0, \pi/\bar{s}) \cap \overline{\mathcal{V}}$.

(2) Further, suppose that Assumption 6 holds. Then: (s_0, q_0) is identified as the unique solution to the minimum distance criterion function based on Eq. (24) evaluated on $\mathcal{U}$.

See the Appendix for details of the proof of Theorem 4. In Theorem 4 part (2), the minimum distance criterion function can be constructed similar to Eq. (20).

3.2. Model when $\{I_t\}$ has general discrete support

We now relax Assumption 5 to allow for more general support of the latent $\{I_t\}$.

Assumption 7. $\{I_t\}$ may take values in $\{-k_1, \ldots, 0, \ldots, +k_2\}$, and $\Pr(I_t = -k_1) > 0$, $\Pr(I_t = +k_2) > 0$.

Here, k_1 and k_2 are positive integers, measuring the strength of the order flow. Assumption 7 allows for $\Pr(I_t = 0) = 0$ or $\Pr(I_t = 0) > 0$. It also allows for asymmetric support in the sense that $k_1 \neq k_2$.

Let $\pi_0 = [\vec{\pi}_{0l}]$ denote the unknown true marginal probability distribution of $\{I_t\}$, where $\pi_{0l} := \Pr(I_t = l) \geq 0$, for $l = -k_1, \ldots, 0, \ldots, +k_2$ and $\sum_l \pi_{0l} = 1$. Let $\varphi_{\pi_0}(u) := \mathbb{E}_{\pi_0}[\exp(iuI_t)]$ denote the true c.f. of I_t corresponding to probability π_0, that is $\varphi_{\pi_0}(\cdot) \equiv \varphi_I(\cdot)$, which is analytic and is uniquely determined by the unknown π_0. Denote

$$R(u,u'; s, \pi) := \frac{\varphi_\pi\left((u'-u)\frac{s}{2}\right)}{\varphi_\pi\left(-u\frac{s}{2}\right)\varphi_\pi\left(u'\frac{s}{2}\right)} \quad \text{for any } s \in \mathcal{S} \text{ and } \pi \in \Pi,$$

where $\Pi := \{\pi = [\vec{\pi}_l]$, a probability mass function of I_t satisfying Assumption 7$\}$, and $\varphi_\pi(u) := \mathbb{E}_\pi[\exp(iuI_t)]$ is the c.f. of $\pi \in \Pi$.

Eq. (21) (or (22)) and Assumption 7 imply the following relation:

$$H(u,u') = R(u,u'; s_0, \pi_0) \quad \text{for all } (u,u') \in \overline{\mathcal{V}}^2 . \tag{25}$$

We prove in the Appendix that Eq. (25) identifies both s_0 and π_0.

Theorem 5. *Let Assumptions 1(i), 3 and 7 hold. Then: $s_0 \in \mathcal{S}$ and $\pi_0 \in \Pi$ are identified.*

Recently Zhang and Hodges (2012) consider a model where our Assumption 7 is replaced by $\{I_t\}$ having support in $\{-\lambda, -1, 1, \lambda\}$. They do not study the identification issue but directly apply Bayesian Gibbs method to estimation under the additional assumption of $\varepsilon_t \sim$ i.i.d.$N(0, \sigma_\varepsilon^2)$.

Remark 1. Theorem 5 is more general than Theorem 4, which in turn includes Theorem 3 as a special case. Theorem 5 suggests a natural minimum distance estimation procedure for s_0 and π_0. Let $\widehat{H}(u,u')$ denote a nonparametric consistent estimator of $H(u,u')$ defined in (15), which could be based on the empirical joint characteristic function $\widehat{\varphi}_{\Delta p,2}(u,u')$ of $\varphi_{\Delta p,2}(u,u')$ defined in (5). Then one could estimate (s_0, π_0) by $(\widehat{s}, \widehat{\pi})$, where

$$(\widehat{s}, \widehat{\pi}) = \arg \inf_{s \in \mathcal{S}, \pi \in \Pi} \sum_{(u,u') \in \overline{\mathcal{V}}^2} \left|\widehat{H}(u,u') - R(u,u'; s, \pi)\right|^2 .$$

One could then use the Wald statistic based on $\widehat{\pi}$ to test whether Assumption 1(ii) (balanced order flow) holds or not. See Chen et al. (2017) for details.

4. Models with serially dependent $\{I_t\}$

This section presents identification results for extended Roll models that relax both Assumptions 1(ii) and 3(iii) imposed in Section 2. Precisely we assume

Assumption 8. $\{I_t\}_{t=1}^\infty$ is an irreducible and aperiodic first-order Markov chain with an unknown true transition probability matrix $\mathcal{Q}_0 := [q_{j,m}^0]$ where

$$q_{j,m}^0 := \Pr(I_t = m | I_{t-1} = j) \quad \text{for } j, m = -k, \ldots, 0, \ldots, +k, \quad \text{and} \quad \sum_m q_{j,m}^0 = 1. \tag{26}$$

Therefore I_t is no longer sub-independent of $-I_{t-1}$ and Assumption 2 is no longer satisfied, and hence Theorem 1 is no longer applicable. Nevertheless, we shall establish the joint identification of $\varphi_\varepsilon(\cdot)$ and s_0 under Assumptions 1(i) and 3(i)(ii) and 8.

Let $\pi_0 = [\vec{\pi}_{0l}]$ denote the unknown true marginal probability distribution of $\{I_t\}$, where $\pi_{0l} := \Pr(I_t = l)$ for $l = -k, \ldots, 0, \ldots, +k$ and $\sum_l \pi_{0l} = 1$. Let P_0 denote the unknown true joint probability distribution of (I_t, I_{t-1}). Under Assumption 8, $\{I_t\}_{t=1}^\infty$ is an ergodic finite-state Markov chain, therefore $\pi_{0l} > 0$ for $l = -k, \ldots, 0, \ldots, +k$ and $\mathcal{Q}_0$ uniquely determines π_0 and P_0 (see, e.g., Definition 4.2.7 and Theorem 4.3.1 of Gallager (2014)).

Under Assumptions 1(i) and 3(i)(ii), we have: for all $(u, u') \in \mathbb{R}^2$,

$$\varphi_{\Delta p,2}(u,u') = \varphi_\varepsilon(u)\varphi_\varepsilon(u')\mathbb{E}\left(\exp\left[iu\frac{s_0}{2}(I_t - I_{t-1})\right] \times \exp\left[iu'\frac{s_0}{2}(I_{t-1} - I_{t-2})\right]\right). \quad (27)$$

This and Assumption 8 together yield the following identification relation

$$H(u,u') = R(u,u';s_0,P_0) \qquad \text{for all } (u,u') \in \overline{\mathcal{V}}^2, \quad (28)$$

where

$$R(u,u';s_0,P_0) := \frac{\mathbb{E}\left(\exp\left[iu\frac{s_0}{2}(I_t - I_{t-1})\right]\exp\left[iu'\frac{s_0}{2}(I_{t-1} - I_{t-2})\right]\right)}{\mathbb{E}\left(\exp\left[iu\frac{s_0}{2}(I_t - I_{t-1})\right]\right)\mathbb{E}\left(\exp\left[iu'\frac{s_0}{2}(I_{t-1} - I_{t-2})\right]\right)}.$$

Under Assumption 8, the support of $(I_t - I_{t-1})$ is $\{-2k, \ldots, 0, \ldots, +2k\}$, and the joint support of $(I_{t-1} - I_{t-2}, I_t - I_{t-1})$ is given in expression in Box II in the Appendix. Let $\mathcal{Q}^0_{\Delta I}$ denote the joint probability mass matrix of $(I_{t-1} - I_{t-2}, I_t - I_{t-1})$, which is a $(4k+1) \times (4k+1)$ matrix. Let $B_{\mathcal{Q}_0}$ be a $(2k+1) \times (4k+1)$ matrix whose entries are either zero or simple functions of $Q^0_{j,\circ} = [q^0_{j,-k}, \ldots, q^0_{j,k}]$ (the jth row vector of $\mathcal{Q}_0$) for $j = -k, \ldots, 0, \ldots, +k$. Let $A_{\mathcal{Q}_0,\pi_0}$ denote a $(4k+1) \times (2k+1)$ matrix whose entries are either zeros or simple products $\pi_{0l}q^0_{i,j}$ for $l, i, j = -k, \ldots, 0, \ldots, +k$. See the Appendix for the precise expressions of $A_{\mathcal{Q}_0,\pi_0}$ and $B_{\mathcal{Q}_0}$. The following equation shows the relation between $\mathcal{Q}^0_{\Delta I}$ and $\mathcal{Q}_0, \pi_0$:

$$\mathcal{Q}^0_{\Delta I} = A_{\mathcal{Q}_0,\pi_0} \times B_{\mathcal{Q}_0}. \quad (29)$$

Therefore the rank of $\mathcal{Q}^0_{\Delta I}$ is at most $2k+1$.

Let $\mathcal{P}_{all}$ be the set of possible joint probability measures P of (I_t, I_{t-1}) satisfying Assumption 8. Let $A_{\mathcal{Q},\pi}$ (defined in the Appendix) be a $(4k+1) \times (2k+1)$ matrix associated with a $P \in \mathcal{P}_{all}$. Define

$$\mathcal{P} := \left\{P \in \mathcal{P}_{all} : A_{\mathcal{Q},\pi} \text{ has full column rank } 2k+1; \; q_{-k,-k} > \frac{1}{2},\; q_{k,k} > \frac{1}{2}\right\}. \quad (30)$$

Assumption 9. (i) $s_0 \in \mathcal{S}$; (ii) $P_0 \in \mathcal{P}$.

Given the expression for $A_{\mathcal{Q}_0,\pi_0}$ in the Appendix, it being of full column rank is easily satisfied. For example, if $q^0_{k,j} > 0$, for $j = -k, \ldots, k$, or $q^0_{-k,j} > 0$, for $j = -k, \ldots, k$, then $A_{\mathcal{Q}_0,\pi_0}$ is of full column rank. Also, when $k = 1$, the assumption that $q^0_{-k,-k} > \frac{1}{2}$ and $q^0_{k,k} > \frac{1}{2}$ could be interpreted as a model of (time-varying) autocorrelation in the trade indicators: after a buy, the most likely thing is another buy, and analogously for a sell.

Let $\varphi_{\Delta I}(\cdot,\cdot)$ denote the true unknown joint c.f. of $(I_{t-1} - I_{t-2}, I_t - I_{t-1})$. We note that the identification of $\mathcal{Q}^0_{\Delta I}$ is equivalent to the identification of $\varphi_{\Delta I}(\cdot,\cdot)$. We establish the following identification results in the Appendix.

Theorem 6. *Let Assumptions 1(i), 3(i)(ii), 8 and 9 hold. Then:*

(1) $(s_0, \varphi_{\Delta I}(\cdot,\cdot))$ *are identified; and* φ_ε *is identified as* $\varphi_\varepsilon(u) = \varphi_{\Delta p,1}(u)[\varphi_{\Delta I}(\frac{s_0}{2}u, 0)]^{-1}$ *on* $\overline{\mathcal{V}}$.

(2) *If, in addition,* $q^0_{k,-j} > 0$ *for* $j = 1, \ldots, k$ *and* $q^0_{-k,j} > 0$ *for* $j = 0, 1, \ldots, k$, *then the joint distribution* P_0 *of* (I_{t-1}, I_t) *is identified.*

Theorem 6 Part (1) establishes the identification of $\mathcal{Q}^0_{\Delta I}$. Then $B_{\mathcal{Q}_0}$ or equivalently the joint distribution P_0 of (I_{t-1}, I_t) can be recovered from the relation $\mathcal{Q}^0_{\Delta I} = A_{\mathcal{Q}_0,\pi_0} \times B_{\mathcal{Q}_0}$ under some conditions on $A_{\mathcal{Q}_0,\pi_0}$. Theorem 6 part (2) provides one such sufficient condition. Note that under Assumption 8, $q^0_{k,-j} > 0$ for $j = 1, \ldots, k$ and $q^0_{-k,j} > 0$ for $j = 0, 1, \ldots, k$ imply that $A_{\mathcal{Q}_0,\pi_0}$ has full column rank. Also, when $k = 1$, the assumption that $q^0_{1,-1} > 0$, $q^0_{-1,0} > 0$ and $q^0_{-1,1} > 0$ is natural.

This problem is related to but cannot be directly implied by the existing identification results for a hidden Markov model with time series data alone. Recently Gassiat and Rousseau (2016) considers identification in a hidden Markov time series model under the assumption that the transition probability matrix is of full rank (see their Theorem 1). From Eq. (29) we note that $\mathcal{Q}^0_{\Delta I}$ in our model fails to satisfy their full rank condition. Since we only have a single time series observation $\{p_t\}$, our identification results cannot be derived from the existing results (e.g., Hu and Shum (2012) and Hu (forthcoming) and the references therein) on hidden Markov panel data models with a large independent cross-section but a fixed finite time period, either.

5. Adverse selection

We have assumed that the price dynamics follow Eq. (1) (Assumption 1(i)) in all the extensions in Sections 3 and 4. We now relax this condition to allow for adverse selection problems.

We relax Eq. (1) and suppose that

$$p_t = p^*_t + I_t\frac{s_0}{2}, \quad p^*_t = p^*_{t-1} + \delta I_t + \varepsilon_t. \quad (31)$$

This equation arises from considering the presence of an adverse selection component in the spread, see Eq. (5.4) in Foucault et al. (2013). Here, δ measures the contribution of adverse selection, i.e., the effect of a market order on the latent true efficient price. This implies that

$$\Delta p_t = \varepsilon_t + \alpha_0 I_t - \beta_0 I_{t-1}, \;\; \text{with } \alpha_0 \equiv \frac{s_0}{2} + \delta, \;\; \beta_0 \equiv \frac{s_0}{2}. \quad (32)$$

Rewriting (32) in the form of our previous price dynamics in (2), i.e., $\Delta p_t = \Delta p^*_t + (I_t - I_{t-1})s_0/2$, we have $\Delta p^*_t = \varepsilon_t + \delta I_t$, and thus $\mathrm{Cov}\left(\Delta p^*_t, I_t\right) = \delta Var(I_t) \neq 0$ whenever $\delta \neq 0$. Hence the Roll and Hasbrouck spread estimators would be inconsistent (i.e., biased even as sample size goes to infinity). If $\{p_t\}$ is the only observable, even assuming $\varepsilon_t \sim$ i.i.d. $N(0, \sigma^2_\varepsilon)$ as in Hasbrouck (2004), $(\alpha_0, \beta_0, \sigma^2_\varepsilon)$ is still not jointly identified. We now show how to regain identification by slightly strengthening Assumption 3 to Assumption 10(ii) below. [8]

Assumption 10. (i) Data $\{p_t\}_{t=1}^T$ is generated from Eq. (32) with $\alpha_0 \neq 0$ and $\beta_0 > 0$, where $\{\varepsilon_t, I_t\}_{t=1}^\infty$ is a strictly stationary process; and (ii) $\varepsilon_t, \varepsilon_{t-1}, I_t, I_{t-1}$ and I_{t-2} are independent.

Assumption 10 implies that for all $(u, u') \in \mathbb{R}^2$,

$$\varphi_{\Delta p,2}(u,u') = \varphi_\varepsilon(u)\varphi_\varepsilon(u')\varphi_I(u\alpha_0)\varphi_I(u'\alpha_0 - u\beta_0)\varphi_I(-u'\beta_0), \quad (33)$$

$$\varphi_{\Delta p,1}(u) = \varphi_{\Delta p,2}(u,0) = \varphi_\varepsilon(u)\varphi_I(u\alpha_0)\varphi_I(-u\beta_0). \quad (34)$$

Eq. (34) immediately implies that the c.f. $\varphi_\varepsilon(\cdot)$ is identified once after (α_0, β_0) and $\varphi_I(\cdot)$ are identified. Also Eq. (32) implies that the identification of (s_0, δ) is equivalent to the identification of (α_0, β_0) via the relation $s_0 = 2\beta_0$ and $\delta = \alpha_0 - \beta_0$.

Eq. (33) also implies

$$H(u,u') = \frac{\varphi_I\left(u'\alpha_0 - u\beta_0\right)}{\varphi_I\left(-u\beta_0\right)\varphi_I\left(u'\alpha_0\right)} \qquad \text{for all } (u,u') \in \overline{\mathcal{V}}^2. \quad (35)$$

Since I_t is a discrete random variable, $\varphi_I(\cdot)$ is analytic, and hence $H(u,u')$ is analytic in $(u,u') \in \overline{\mathcal{V}}^2$. Relation (35) immediately implies that

$$\frac{\partial^2 H(0,0)}{\partial u \partial u'} = \alpha_0\beta_0 Var(I_t), \quad (36)$$

[8] Instead of imposing Assumption 10(ii), we could also obtain the identification and consistent estimation of (α_0, β_0) when additional data such as trade volume is available.

hence the sign of α_0 is identified as the sign of $\frac{\partial^2 H(0,0)}{\partial u \partial u'}$. Therefore in the rest of this section we could assume that $0 < \beta_0 \in \mathcal{B} := [0, \bar{b}]$ and $0 \neq \alpha_0 \in \mathcal{B}_1 := [-\bar{b}, \bar{b}]$ for some finite $\bar{b} \geq \bar{s}/2$.

In the next several subsections we present the identification of (α_0, β_0) when the functional form of $\varphi_I(\cdot)$ is completely known, known up to a unknown parameter, or unknown.

5.1. Adverse selection with balanced order flow

Under Assumption 1(ii) (balanced order flow), $\varphi_I(u) = \cos(u)$ for all $u \in \mathbb{R}$ and $Var(I_t) = 1$. Denote

$$\overline{\mathcal{U}}_{as} := \left\{ (u, u') \in \overline{\mathcal{V}}^2 : \min_{\alpha \in \mathcal{B}_1, \beta \in \mathcal{B}} |\cos(u\beta) \cos(u'\alpha)| > 0 \right\}, \tag{37}$$

and a function on $\overline{\mathcal{U}}_{as} \times \mathcal{B}_1 \times \mathcal{B}$ as

$$R(u, u'; \alpha, \beta) := \frac{\cos(u'\alpha - u\beta)}{\cos(u\beta)\cos(u'\alpha)} = 1 + \frac{\sin(u\beta)\sin(u'\alpha)}{\cos(u\beta)\cos(u'\alpha)}.$$

Eq. (35) and Assumption 1(ii) now imply that

$$H(u, u') = R(u, u'; \alpha_0, \beta_0) \ \text{for all } (u, u') \in \overline{\mathcal{V}}^2. \tag{38}$$

Since $\overline{\mathcal{V}}^2$ contains an open ball of $(0, 0)$, for a small positive $\tilde{u} \in \overline{\mathcal{V}}$, we have $(\tilde{u}, \tilde{u}), (\tilde{u}, 2\tilde{u}), (2\tilde{u}, \tilde{u}) \in \overline{\mathcal{V}}^2$, and Eq. (38) yields

$$\begin{aligned} \sin^2(\tilde{u}\alpha_0) &= \frac{2H(\tilde{u}, \tilde{u}) - H(\tilde{u}, 2\tilde{u}) - 1}{2H(\tilde{u}, \tilde{u}) - 2H(\tilde{u}, 2\tilde{u})}, \\ \sin^2(\tilde{u}\beta_0) &= \frac{2H(\tilde{u}, \tilde{u}) - H(2\tilde{u}, \tilde{u}) - 1}{2H(\tilde{u}, \tilde{u}) - 2H(2\tilde{u}, \tilde{u})}. \end{aligned} \tag{39}$$

Assumption 11. (i) $(\alpha_0, \beta_0) \in \mathcal{B}_1 \times \mathcal{B}$; (ii) either (a) $\mathcal{U} = \overline{\mathcal{U}}_{as}$; or (b) $\mathcal{U} \subset \overline{\mathcal{U}}_{as}$ and $\exists (\tilde{u}, \tilde{u}), (\tilde{u}, 2\tilde{u}), (2\tilde{u}, \tilde{u}) \in \mathcal{U}$ such that $\tilde{u} \in (0, \frac{\pi}{2\bar{b}})$.

For any $\tilde{u} \in (0, \frac{\pi}{2\bar{b}})$, $a \mapsto \sin^2(\tilde{u}a)$ is strictly increasing in $a \in \mathcal{B} = [0, \bar{b}]$. Hence Eq. (39) can be used to solve $|\alpha_0| \in \mathcal{B}$ and $\beta_0 \in \mathcal{B}$ uniquely as

$$\begin{aligned} |\alpha_0| &= \tilde{u}^{-1} \arcsin\left(\sqrt{\frac{2H(\tilde{u}, \tilde{u}) - H(\tilde{u}, 2\tilde{u}) - 1}{2H(\tilde{u}, \tilde{u}) - 2H(\tilde{u}, 2\tilde{u})}} \right), \\ \beta_0 &= \tilde{u}^{-1} \arcsin\left(\sqrt{\frac{2H(\tilde{u}, \tilde{u}) - H(2\tilde{u}, \tilde{u}) - 1}{2H(\tilde{u}, \tilde{u}) - 2H(2\tilde{u}, \tilde{u})}} \right). \end{aligned} \tag{40}$$

We are ready to state the following results.

Theorem 7. *Let Assumption 1(ii), 10 and 11(i) hold. Then:*

(1) *(α_0, β_0) is identified by Eqs. (36) and (40) with some $\tilde{u} \in (0, \frac{\pi}{2\bar{b}}) \cap \overline{\mathcal{V}}$, and φ_ε is identified on $\overline{\mathcal{V}}$ as $\varphi_\varepsilon(u) = \varphi_{\Delta p,1}(u)[\cos(u\alpha_0)\cos(u\beta_0)]^{-1}$.*

(2) *Further, let Assumption 11 hold. Then: (α_0, β_0) is identified as the unique solution to the minimum distance criterion function based on Eq. (38) evaluated on $\mathcal{U}$.*

In Theorem 7 part (2), the minimum distance criterion function can be constructed similar to Eq. (20).

5.2. Adverse selection with unbalanced order flow

Under Assumption 5, $\varphi_I(u) = \cos(u) + i(2q_0 - 1)\sin(u)$ for all $u \in \mathbb{R}$, for a unknown $q_0 \in (0, 1)$.

Denote a function on $\overline{\mathcal{U}}_{as} \times \mathcal{B}_1 \times \mathcal{B} \times (0, 1)$ as

$$R(u, u'; \alpha, \beta, q) := \frac{\left(1 + \tan(u'\alpha)\tan(u\beta)\right) + i(2q - 1)\left(\tan(u'\alpha) - \tan(u\beta)\right)}{[1 - i(2q - 1)\tan(u\beta)][1 + i(2q - 1)\tan(u'\alpha)]}.$$

Eq. (33) (or (35)) and Assumption 5 now imply that

$$H(u, u') = R(u, u'; \alpha_0, \beta_0, q_0) \ \text{for all } (u, u') \in \overline{\mathcal{V}}^2. \tag{41}$$

We establish the following result in the Appendix.

Theorem 8. *Let Assumptions 5, 10 and 11(i) hold. Then: (α_0, β_0, q_0) is identified by Eqs. (78), (74) and (77) in the Appendix with some $\tilde{u} \in (0, \frac{\pi}{2\bar{b}}) \cap \overline{\mathcal{V}}$; and φ_ε is identified on $\overline{\mathcal{V}}$ as : $\varphi_\varepsilon(u) = \varphi_{\Delta p,1}(u)([\cos(u\alpha_0) + i(2q_0 - 1)\sin(u\alpha_0)][\cos(u\beta_0) - i(2q_0 - 1)\sin(u\beta_0)])^{-1}$.*

Theorem 8 becomes Theorem 7 part (1) when $q_0 = 1/2$.

5.3. Adverse selection when $\{I_t\}$ has general discrete support

We now relax Assumption 5 to Assumption 7, and the c.f. $\varphi_I(\cdot)$ becomes a unknown analytic function. Many notations and definitions in this subsection are the same as those in Section 3.2. Recall that π_0 denotes the unknown true marginal probability distribution of $\{I_t\}$, and $\varphi_{\pi_0}(\cdot) = \varphi_I(\cdot)$ denotes the true c.f. of I_t corresponding to probability π_0. Denote

$$R(u, u'; \alpha, \beta, \pi) := \frac{\varphi_\pi(u'\alpha - u\beta)}{\varphi_\pi(-u\beta)\varphi_\pi(u'\alpha)},$$

for any $(\alpha, \beta) \in \mathcal{B}_1 \times \mathcal{B}$ and $\pi \in \Pi$. And $\varphi_\pi(u) := \mathbb{E}_\pi[\exp(iuI_t)]$ is the c.f. of $\pi \in \Pi$.

Eq. (33) (or (35)) and Assumption 7 now imply the following relation:

$$H(u, u') = R(u, u'; \alpha_0, \beta_0, \pi_0) \qquad \text{for all } (u, u') \in \overline{\mathcal{V}}^2. \tag{42}$$

We prove in the Appendix that Eq. (42) identifies both (α_0, β_0) and π_0.

Theorem 9. *Let Assumptions 7, 10 and 11(i) hold. Then: (α_0, β_0) and $\pi_0 \in \Pi$ are identified; and φ_ε is identified on $\overline{\mathcal{V}}$ as : $\varphi_\varepsilon(u) = \varphi_{\Delta p,1}(u)[\varphi_I(u\alpha_0)\varphi_I(-u\beta_0)]^{-1}$.*

Remark 2. Theorem 9 is more general than Theorem 8, except that (α_0, β_0, q_0) could be solved in closed form in Theorem 8. Theorem 9 suggests a natural minimum distance estimation procedure for (α_0, β_0) and π_0. Let $\widehat{H}(u, u')$ denote a nonparametric consistent estimator of $H(u, u')$ as in Remark 1. Then one could estimate $(\alpha_0, \beta_0, \pi_0)$ by $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi})$, where

$$(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}) = \arg \inf_{\alpha \in \mathcal{B}_1, \beta \in \mathcal{B}, \pi \in \Pi} \sum_{(u,u') \in \overline{\mathcal{V}}^2} |\widehat{H}(u, u') - R(u, u'; \alpha, \beta, \pi)|^2.$$

One could then use a Wald statistic to test $\alpha_0 = \beta_0$ (no adverse selection), regardless whether Assumption 1(ii) holds or not. See Chen et al. (2017) for details.

6. Random spread

Consider the model with a random spread:

$$\begin{aligned} p_t &= p_t^* + \frac{s_t}{2} I_t, \quad p_t^* = p_{t-1}^* + \varepsilon_t, \\ \Delta p_t &= \varepsilon_t + \frac{1}{2}\left(s_t I_t - s_{t-1} I_{t-1}\right). \end{aligned} \tag{43}$$

Assumption 12. (i) Data $\{p_t\}_{t=1}^T$ is generated from Eq. (43), where $\{\varepsilon_t, s_t I_t\}_{t=1}^\infty$ is a strictly stationary process; (ii) I_t is independent of s_t, and Assumptions 1(ii) holds; (iii) ε_t is sub-independent of $(s_t I_t - s_{t-1} I_{t-1})/2$; $s_t I_t$ is sub-independent of $-s_{t-1} I_{t-1}$; (iv) $\varepsilon_t + \varepsilon_{t-1}$ is sub-independent of $(s_t I_t - s_{t-2} I_{t-2})/2$; $s_t I_t$ is sub-independent of $-s_{t-2} I_{t-2}$; and ε_t is sub-independent of ε_{t-1}.

Assumption 12(i)(ii) is a natural extension of Assumption 1. Assumption 12(iii)(iv) is a natural extension of Assumption 2.

Under Assumption 12, we have for all $u \in \mathbb{R}$,

$$\varphi_{\Delta p,1}(u) = \varphi_\varepsilon(u)\left(\mathbb{E}\left[\cos\left(u\frac{s_t}{2}\right)\right]\right)^2, \qquad \varphi_{\Delta^2 p}(u) = \varphi_\varepsilon^2(u)\left(\mathbb{E}\left[\cos\left(u\frac{s_t}{2}\right)\right]\right)^2. \tag{44}$$

This immediately implies that the c.f. $\varphi_\varepsilon(\cdot)$ is identified as (9) on $\overline{\mathcal{V}}$. Next, for $h(\cdot)$ defined in (10), Eq. (44) implies the following relation:

$$h(u) = \left(\mathbb{E}\left[\cos\left(u\frac{s_t}{2}\right)\right]\right)^2 \quad \text{for all } u \in \overline{\mathcal{V}}. \tag{45}$$

Under Assumption 12(i)(ii), $\{s_t\}$ has the same marginal distributions. The next assumption is similar to the condition $s_0 \in (0, \bar{s}]$ for the non-random spread s_0 in all the previous sections.

Assumption 13. The unknown true probability distribution $F_s(\cdot)$ of s_t has support $\mathcal{S} = [0, \bar{s}]$ with $F_s(0) = 0$.

Note that the random spread s_t could be a discrete, or partly discrete and partly continuous random variable since its distribution $F_s()$ is not assumed to be differentiable or strictly increasing. This assumption is extremely mild and reasonable for financial applications.

We prove in the Appendix that Eq. (44) and Assumption 13 together identify the distribution function $F_s(\cdot)$ of the random spread s_t.

Theorem 10. *Let Assumption 12 hold. Then:*

(1) The c.f. $\varphi_\varepsilon(\cdot)$ is identified as (9) on $\overline{\mathcal{V}}$.

(2) If further, Assumption 13 holds, then $F_s(\cdot)$ is identified by Eq. (85) in the Appendix.

7. Multivariate Roll models

Let $p_t = (p_{1,t}, \ldots, p_{n,t})^\intercal \in \mathbb{R}^n$, $I_t = (I_{1,t}, \ldots, I_{n,t})^\intercal \in \{-1, 1\}^n$, $\varepsilon_t = (\varepsilon_{1,t}, \ldots, \varepsilon_{n,t})^\intercal \in \mathbb{R}^n$ and

$$\Delta p_t = \varepsilon_t + \frac{1}{2} S_0 \Delta I_t, \quad \text{where} \quad S_0 = Diag\{s_{1,0}, \ldots, s_{n,0}\}. \tag{46}$$

By applying the identification results of previous sections, each $s_{j,0}$ can be identified using individual price series $\{p_{j,t}\}$ for $j = 1, \ldots, n$. We focus on the identification of the contemporaneous dependence of I_t. For simplicity we consider a simple multivariate extension of the basic Roll model.

Assumption 14. (i) Data $\{p_t\}_{t=1}^T$ is generated from Eq. (46) with $s_{j,0} \in (0, \bar{s}]$ for $j = 1, \ldots, n$ and some finite $\bar{s}$, and $\{\varepsilon_t, I_t\}_{t=1}^\infty$ is a strictly stationary process; (ii) $(\varepsilon_t, \varepsilon_{t-1})$ is independent of $(\Delta I_t, \Delta I_{t-1})$; (iii) ε_t is independent of ε_{t-1}; and (iv) I_t, I_{t-1} and I_{t-2} are independent.

This assumption implies that for any $(u_1, u_2) \in \mathbb{R}^{2n}$,

$$\begin{aligned}\varphi_{\Delta p,2}(u_1, u_2) &:= \mathbb{E}\left(\exp\left(iu_1^\intercal \Delta p_t + iu_2^\intercal \Delta p_{t-1}\right)\right) \\ &= \varphi_\varepsilon(u_1)\varphi_\varepsilon(u_2)\mathbb{E}\left(\exp\left(\frac{i}{2}u_1^\intercal S_0 I_t\right)\right) \\ &\quad \times \mathbb{E}\left(\exp\left(\frac{i}{2}(u_2 - u_1)^\intercal S_0 I_{t-1}\right)\right) \\ &\quad \times \mathbb{E}\left(\exp\left(-\frac{i}{2}u_2^\intercal S_0 I_{t-2}\right)\right).\end{aligned} \tag{47}$$

Eq. (47) evaluated at any $(u, 0) \in \mathbb{R}^{2n}$ yields the relation for the c.f. of Δp_t:

$$\begin{aligned}\varphi_{\Delta p,1}(u) := \varphi_{\Delta p,2}(u, 0) &= \varphi_\varepsilon(u)\mathbb{E}\left(\exp\left(\frac{i}{2}u^\intercal S_0 I_t\right)\right) \\ &\quad \times \mathbb{E}\left(\exp\left(-\frac{i}{2}u^\intercal S_0 I_{t-1}\right)\right).\end{aligned} \tag{48}$$

Let $\overline{\mathcal{W}} := \{u \in \mathbb{R}^n : \varphi_{\Delta p,1}(u) \neq 0\}$, which contains an open ball of $0 \in \mathbb{R}^n$. Eqs. (47) and (48) immediately imply the identification of the c.f. $\varphi_\varepsilon(u)$ on $\overline{\mathcal{W}}$, and for all $(u_1, u_2) \in \overline{\mathcal{W}}^2$,

$$\begin{aligned}H(u_1, u_2) &:= \frac{\varphi_{\Delta p,2}(u_1, u_2)}{\varphi_{\Delta p,1}(u_1)\varphi_{\Delta p,1}(u_2)} \\ &= \frac{\mathbb{E}\left(\exp\left(\frac{i}{2}(u_2 - u_1)^\intercal S_0 I_{t-1}\right)\right)}{\mathbb{E}\left(\exp\left(-\frac{i}{2}u_1^\intercal S_0 I_{t-1}\right)\right)\mathbb{E}\left(\exp\left(\frac{i}{2}u_2^\intercal S_0 I_{t-1}\right)\right)}.\end{aligned} \tag{49}$$

The next assumption imposes a structure on the contemporaneous dependence of I_t.

Assumption 15. Let Ω be a symmetric, positive semi-definite $n \times n$ matrix. The diagonal elements of Ω equal to one and the off-diagonal elements of Ω are $\{\omega_{jk}\}$.

$$Y_t^* = \left(Y_{1,t}^*, \ldots, Y_{n,t}^*\right)^\intercal \sim N(0, \Omega)$$
$$I_{j,t} = 1\left(Y_{j,t}^* > 0\right) - 1\left(Y_{j,t}^* < 0\right), \quad \textit{for } j = 1, \ldots, n.$$

The covariance matrix Ω is allowed to be singular. For example, when $n = 2$, ω_{12} is allowed to be 1, meaning $I_{1,t} = I_{2,t}$. There are $n(n-1)/2$ free parameters $\{\omega_{jk}\}$ to be identified. For $j \neq k$ we define:

$$\begin{aligned}q_{jk} &:= \Pr\left(I_{j,t} = -1, I_{k,t} = -1\right) = \Pr\left(Y_{j,t}^* < 0, Y_{k,t}^* < 0\right) \\ &:= g\left(\omega_{jk}\right),\end{aligned} \tag{50}$$

where, under Assumption 15, $g(\cdot)$ is strictly increasing. We prove the following result in the Appendix.

Theorem 11. (1) *Let Assumption 14 hold, then $\varphi_\varepsilon(u) = \varphi_{\Delta p,2}(u, u)[\varphi_{\Delta p,1}(u)]^{-1}$ on $\overline{\mathcal{W}}$.*

(2) *Let Assumptions 14 and 15 hold. Then: $s_{j,0}$, $j = 1, \ldots, n$, is identified as in Theorem 2 part (1); q_{jk} is identified as Eq. (86) in the Appendix, and ω_{jk} is identified as $g^{-1}\left(q_{jk}\right)$, for $j, k = 1, \ldots, n$ and $j \neq k$.*

8. Conclusions

In this paper we provide identification of the spread s_0 and the distribution of the latent fundamental price increments ε_t using transaction price time series observations alone. Our identification results do not require the existence of any finite moments of the observed price increments, do not require the full independence between $\{\varepsilon_t\}$ and the latent trade direction indicators $\{I_t\}$, and allow the latent ε_t to be discrete or continuous, symmetric or asymmetric. We first provide closed-form identification results under a mild sub-independence condition in the basic Roll (1984) model. We then establish identification in various extended Roll models, such as models with general unbalanced order flow, or serially dependent latent trade indicators, or adverse selection or a possibly random spread. Identification in a multivariate Roll model is also provided. Our results on the identification of $(s_0, \varphi_\varepsilon)$ and the additional parameters in extended models are established under conditions much weaker than those in the existing literature and are very reasonable for financial applications.

This paper focuses on constructive identification results in basic Roll (1984) and extended Roll models. However, our identification strategy, the minimum distance between the nonparametrically identified (from data) joint characteristic function of

$$R(u,u';s,q):=\frac{\left[\cos\left(u\frac{s}{2}\right)+(2q-1)i\sin\left(u\frac{s}{2}\right)\right]\left[\cos\left(u'\frac{s}{2}\right)-(2q-1)i\sin\left(u'\frac{s}{2}\right)\right]\times\left[\cos\left((u'-u)\frac{s}{2}\right)+(2q-1)i\sin\left((u'-u)\frac{s}{2}\right)\right]}{\left[\cos^2\left(u\frac{s}{2}\right)+(2q-1)^2\sin^2\left(u\frac{s}{2}\right)\right]\left[\cos^2\left(u'\frac{s}{2}\right)+(2q-1)^2\sin^2\left(u'\frac{s}{2}\right)\right]}, \quad (53)$$

Box I.

consecutive one period returns and its model-implied semiparametric counterpart, allows for even more general models that include several features of the extended Roll models all at once. In fact these minimum distance via characteristic functions imply overidentification restrictions in all these models. In the companion paper, Chen et al. (2017), estimation and testing of the Roll type models based on this paper's identification results are presented.

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Appendix A

This Appendix contains additional proofs that are not presented in the main text.

A.1. Additional proofs for Sections 2 and 3

Proof of Theorem 3. The criterion function (20) is nonnegative, with $Q(s_0,\mathcal{U}) = 0$, under Assumption 4(ii). For either case of Assumption 4(ii), $\exists(\tilde u,\tilde u)\in\mathcal{U}$ with $\tilde u>0$. For this grid point, the moment condition (19) yields the relation

$$\cos^2\left(\tilde u\frac{s_0}{2}\right)=\frac{\varphi^2_{\Delta p,1}(\tilde u)}{\varphi_{\Delta p,2}(\tilde u,\tilde u)}. \quad (51)$$

By Assumption 4(ii), $\tilde u$ is smaller than the first positive zero of $u\mapsto \min_{s\in\mathcal{S}}\cos\left(u\frac{s}{2}\right)$, and hence $s\mapsto\cos^2\left(\tilde u\frac{s}{2}\right)$ is strictly decreasing in $s\in\mathcal{S}$. This implies that (51) holds only at $s=s_0\in\mathcal{S}$, which further implies that the criterion function is uniquely minimized at $s=s_0$. Since $Q(s,\mathcal{U})$ is continuous in $s\in\mathcal{S}=[0,\bar s]$, the identifiable uniqueness is trivially satisfied. □

Proof of Theorem 4. Under Assumption 1(i), 3 and 5, we obtain the following special case of Eq. (21): for all $(u,u')\in\mathbb{R}^2$,

$$\varphi_{\Delta p,2}(u,u')=\varphi_\varepsilon(u)\varphi_\varepsilon(u')\left[\cos\left(u\frac{s_0}{2}\right)+(2q_0-1)i\sin\left(u\frac{s_0}{2}\right)\right]\times\left[\cos\left(u'\frac{s_0}{2}\right)-(2q_0-1)i\sin\left(u'\frac{s_0}{2}\right)\right]\times\left[\cos\left((u'-u)\frac{s_0}{2}\right)+(2q_0-1)i\sin\left((u'-u)\frac{s_0}{2}\right)\right]. \quad (52)$$

Hence

$$\varphi_{\Delta p,1}(u)\equiv\varphi_{\Delta p,2}(u,0)=\varphi_\varepsilon(u)\left[\cos^2\left(u\frac{s_0}{2}\right)+(2q_0-1)^2\sin^2\left(u\frac{s_0}{2}\right)\right],$$

$$\varphi_{\Delta^2 p}(u)\equiv\varphi_{\Delta p,2}(u,u)=(\varphi_\varepsilon(u))^2\left[\cos^2\left(u\frac{s_0}{2}\right)+(2q_0-1)^2\sin^2\left(u\frac{s_0}{2}\right)\right].$$

These immediately imply that the c.f. $\varphi_\varepsilon(\cdot)$ is identified as (9) on $\overline{\mathcal{V}}$. In addition to the definitions of $\overline{\mathcal{V}},\overline{\mathcal{U}}$ and $H(u,u')$ given in Section 2, we introduce a function on $\overline{\mathcal{U}}\times\mathcal{S}\times(0,1)$ as (See equation given in Box I.)
which is complex-valued unless $q(q-1)(2q-1)\sin\left(u\frac{s}{2}\right)\sin\left(u'\frac{s}{2}\right)\sin\left((u'-u)\frac{s}{2}\right)=0$. Eq. (24) implies that $H(u,u')$ is complex-valued unless $q_0(q_0-1)(2q_0-1)\sin\left(u\frac{s_0}{2}\right)\sin\left(u'\frac{s_0}{2}\right)\sin\left((u'-u)\frac{s_0}{2}\right)=0$.

For all $(\tilde u,\tilde u)\in\overline{\mathcal{V}}^2$ with $\tilde u\neq0$, the identification Eq. (24) yields the relations

$$H(\tilde u,\tilde u)=\frac{1}{\cos^2\left(\tilde u\frac{s_0}{2}\right)+(2q_0-1)^2\sin^2\left(\tilde u\frac{s_0}{2}\right)},$$
$$\iff\quad \cos^2\left(\tilde u\frac{s_0}{2}\right)=\frac{1/H(\tilde u,\tilde u)-(2q_0-1)^2}{1-(2q_0-1)^2}, \quad (54)$$

where $H(\tilde u,\tilde u)$ is real-valued with $H(\tilde u,\tilde u)>1$. Once $(2q_0-1)^2$ is identified, Eq. (54) can be used to identify s_0 in $\mathcal{S}$ if $\tilde u\in(0,\pi/\bar s)\cap\overline{\mathcal{V}}$ (as in Section 2). For all $(\tilde u,-\tilde u)\in\overline{\mathcal{V}}^2$ with $\tilde u\neq0$, Eq. (24) implies

$$\frac{H(\tilde u,-\tilde u)}{[H(\tilde u,\tilde u)]^2}=\left[\cos\left(\tilde u\frac{s_0}{2}\right)+(2q_0-1)i\sin\left(\tilde u\frac{s_0}{2}\right)\right]^2\times\left[\cos(\tilde u s_0)-(2q_0-1)i\sin(\tilde u s_0)\right].$$

$$\begin{aligned}\mathrm{Re}\left(\frac{H(\tilde u,-\tilde u)}{[H(\tilde u,\tilde u)]^2}\right)&=(2q_0-1)^2+\left[(2q_0-1)^2-1\right]\times\cos^2\left(\tilde u\frac{s_0}{2}\right)\left[1-2\cos^2\left(\tilde u\frac{s_0}{2}\right)\right]\\&=2(2q_0-1)^2-H(\tilde u,\tilde u)^{-1}+2\frac{\left[H(\tilde u,\tilde u)^{-1}-(2q_0-1)^2\right]^2}{1-(2q_0-1)^2},\end{aligned}$$

where the last equality uses the relation implied by Eq. (54). The first derivative of the right-hand side of the above equation with respect to $(2q_0-1)^2$ is equal to $2\frac{\left[H(\tilde u,\tilde u)^{-1}-1\right]^2}{\left[1-(2q_0-1)^2\right]^2}$, which is strictly positive, since $H(\tilde u,\tilde u)>1$ and $q_0\in(0,1)$. Therefore, $(2q_0-1)^2$ can be uniquely identified as

$$(2q_0-1)^2=\frac{\mathrm{Re}\left(\frac{H(\tilde u,-\tilde u)}{[H(\tilde u,\tilde u)]^2}\right)+H(\tilde u,\tilde u)^{-1}-2H(\tilde u,\tilde u)^{-2}}{2+\mathrm{Re}\left(\frac{H(\tilde u,-\tilde u)}{[H(\tilde u,\tilde u)]^2}\right)-3H(\tilde u,\tilde u)^{-1}}. \quad (55)$$

Finally,

$$\begin{aligned}\mathrm{Im}\left(\frac{H(\tilde u,-\tilde u)}{[H(\tilde u,\tilde u)]^2}\right)&=\left[(2q_0-1)^2-1\right](2q_0-1)\sin^2\left(\tilde u\frac{s_0}{2}\right)\sin(\tilde u s_0)\\&=2(1-2q_0)(1-H(\tilde u,\tilde u)^{-1})\times\sqrt{\frac{1/H(\tilde u,\tilde u)-(2q_0-1)^2}{1-(2q_0-1)^2}}\sqrt{\frac{1-1/H(\tilde u,\tilde u)}{1-(2q_0-1)^2}},\end{aligned} \quad (56)$$

$$\begin{bmatrix} & & & & (-2k,0) & \cdots & \cdots & \cdots & (-2k,2k) \\ & & & (-2k+1,-1) & (-2k+1,0) & \cdots & \cdots & \cdots & (-2k+1,2k) \\ & & (-2k+2,-2) & (-2k+2,-1) & (-2k+2,0) & \cdots & \cdots & \cdots & (-2k+2,2k) \\ & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & (-1,-2k+1) & \cdots & \cdots & (-1,0) & \cdots & \cdots & (-1,2k-1) & (-1,2k) \\ (0,-2k) & (0,-2k+1) & \cdots & \cdots & (0,0) & \cdots & & \cdots(0,2k-1) & (0,2k) \\ (1,-2k) & (1,-2k+1) & \cdots & \cdots & (1,0) & \cdots & \cdots & (1,2k-1) & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ (2k-2,-2k) & \cdots & \cdots & \cdots & (2k-2,0) & (2k-2,1) & (2k-2,2) & & \\ (2k-1,-2k) & \cdots & \cdots & \cdots & (2k-1,0) & (2k-1,1) & & & \\ (2k,-2k) & \cdots & \cdots & \cdots & (2k,0) & & & & \end{bmatrix}. \quad (61)$$

Box II.

which can be used to identify the sign of $2q_0 - 1$ for a small $\tilde{u} > 0$. These arguments establish the statements in the theorem. □

Proof of Theorem 5. Under Assumption 1(i) and 3, we obtain Eq. (21) (and Eqs. (6) and (7) with $\varphi_{\pi_0} \equiv \varphi_I$ in Section 2). Hence Theorem 1 remains valid and the c.f. $\varphi_\varepsilon(\cdot)$ is identified as (9) on $\overline{\mathcal{V}}$.

Eq. (21) also implies that, on $\overline{\mathcal{V}}^2$, Eq. (25) is satisfied by the true parameter value $(s_0 \in \mathcal{S}, \varphi_{\pi_0})$. Suppose another pair $(\tilde{s} \in \mathcal{S}, \psi(\cdot))$ also satisfies Eq. (25), where ψ denotes the c.f. associated with another probability mass function π satisfying Assumption 7. That is, on $\overline{\mathcal{V}}^2$ we have:

$$H(u,u') = \frac{\varphi_{\pi_0}\left(\frac{s_0}{2}\left(u'-u\right)\right)}{\varphi_{\pi_0}\left(\frac{s_0}{2}u'\right)\varphi_{\pi_0}\left(-\frac{s_0}{2}u\right)} = \frac{\psi\left(\frac{\tilde{s}}{2}\left(u'-u\right)\right)}{\psi\left(\frac{\tilde{s}}{2}u'\right)\psi\left(-\frac{\tilde{s}}{2}u\right)}. \quad (57)$$

Below we shall prove that, without any restriction on the support of $\{I_t\}$ (such as Assumption 7), $\varphi_{\pi_0}(\frac{s_0}{2}u) = \exp(ifu)\psi\left(\frac{\tilde{s}}{2}u\right)$, where $f \in \mathbb{R}$ is a constant, This result is intuitive. Since we only have observations for $\frac{s_0}{2}(I_t - I_{t-1})$, we could not differentiate between I_t and $I_t + f$, for a constant f, or between (I_t, s_0) and $(I_t \cdot \frac{s_0}{\tilde{s}}, \tilde{s})$, for a positive constant $\tilde{s}$, without additional information about the support. Assumption 7 excludes the possibility of a change of the location or the scale, then $\theta_0 = (s_0, \pi_0^{\intercal})^{\intercal}$ can be uniquely identified from Eq. (25). Denote $h(u) = \psi\left(\frac{\tilde{s}}{s_0}u\right)$, and $u_1 = -\frac{s_0}{2}u$, $u_2 = \frac{s_0}{2}u'$. Note that $\varphi_{\pi_0}(\cdot)$, $\psi(\cdot)$, $h(\cdot)$ are all analytic on $\mathbb{R}$ and equal to 1 at 0. There exists a small neighbourhood $\mathcal{M}$ of $(0,0) \subset \overline{\mathcal{V}}^2$, such that $\varphi_{\pi_0}(u_1)$, $\varphi_{\pi_0}(u_2)$, $\varphi_{\pi_0}(u_1+u_2)$, $h(u_1)$, $h(u_2)$ and $h(u_1+u_2)$ are all bounded away from zero on $(u_1,u_2) \in \mathcal{M}$. Eq. (57) gives

$$\frac{\varphi_{\pi_0}(u_1+u_2)}{h(u_1+u_2)} = \frac{\varphi_{\pi_0}(u_1)}{h(u_1)}\frac{\varphi_{\pi_0}(u_2)}{h(u_2)}. \quad (58)$$

Define $\gamma(u) = \frac{\varphi_{\pi_0}(u)}{h(u)}$, which is analytic on an open interval of 0. Eq. (58) can be rewritten as

$$\gamma(u_1+u_2) = \gamma(u_1)\gamma(u_2). \quad (59)$$

In Theorem 1 on page 38 of Aczel (1966), it has been shown that the only nonzero analytic solutions of (59) are the exponential functions, $\exp(au)$, where $a \in \mathcal{C}$ is a constant. Namely, $\varphi_{\pi_0}(\frac{s_0}{2}u) = \exp(\tilde{a}u)\psi\left(\frac{\tilde{s}}{2}u\right)$, for some fixed $\tilde{a} \in \mathcal{C}$. Since, for all $u \in \mathbb{R}$, $\varphi_{\pi_0}(-\frac{s_0}{2}u) = \overline{\varphi_{\pi_0}(\frac{s_0}{2}u)}$ and $\psi\left(-\frac{\tilde{s}}{2}u\right) = \overline{\psi\left(\frac{\tilde{s}}{2}u\right)}$, it is straightforward to show $\tilde{a} = if$, for some $f \in \mathbb{R}$. Equivalently,

$$\frac{s_0}{2}I_t = \frac{\tilde{s}}{2}\widetilde{I_t} + f, \quad (60)$$

where the c.f. of I_t is $\varphi_{\pi_0}(u)$, and the c.f. of $\widetilde{I_t}$ is $\psi(u)$. Eq. (60) implies the number of points in the support of I_t is also identified. Let the ordered sets $\{m_1, m_2, \ldots, m_l\} \subset \{-k_1, \ldots, 0, \ldots, +k_2\}$ and $\{\widetilde{m}_1, \widetilde{m}_2, \ldots, \widetilde{m}_l\} \subset \{-k_1, \ldots, 0, \ldots, +k_2\}$ denote the supports of I_t and $\widetilde{I}_t$, respectively. Eq. (60) implies, for all $i = 1, \ldots, l$,

$$\widetilde{m}_i = \frac{s_0}{\tilde{s}}m_i - f\frac{2}{\tilde{s}}.$$

Since $m_1 = \widetilde{m}_1 = -k_1$, and $m_l = \widetilde{m}_l = +k_2$, $s_0 = \tilde{s}$ and $f = 0$. Therefore, s_0 and the distribution of I_t can be uniquely identified. □

A.2. Additional proofs for Section 4

Proof of Theorem 6 Part (1). Since $\{I_t\}$ takes values in $\{-k, \ldots, 0, \ldots, +k\}$, the support of $(I_t - I_{t-1})$ is $\{-2k, \ldots, 0, \ldots, +2k\}$ and the joint support of $(I_{t-1} - I_{t-2}, I_t - I_{t-1})$ is given in Box II.

Let $P \in \mathcal{P}$ denote any candidate joint probability distribution of (I_t, I_{t-1}). Let $\pi = [\vec{\pi}_l]$ denote the corresponding marginal probability distribution of $\{I_t\}$, and $\mathcal{Q}$ the corresponding transition probability matrix with j-th row vector being $Q_{j,\circ} = [q_{j,-k}, \ldots, q_{j,k}]$, for $j = -k, \ldots, 0, \ldots, +k$, where the summation of each component of $Q_{j,\circ}$ equals to 1 by definition. Let $\mathcal{Q}_{\Delta I}$ denote the corresponding joint probability mass matrix of $(I_{t-1} - I_{t-2}, I_t - I_{t-1})$, which is a $(4k+1) \times (4k+1)$ matrix. The following equation shows the connection between $\mathcal{Q}_{\Delta I}$ and $\mathcal{Q}, \pi$:

$$\mathcal{Q}_{\Delta I} = A_{\mathcal{Q},\pi} \times B_{\mathcal{Q}}, \quad (62)$$

where $A_{\mathcal{Q},\pi}$ is the $(4k+1) \times (2k+1)$ matrix given in Box III and $B_{\mathcal{Q}}$ is the $(2k+1) \times (4k+1)$ matrix given in Box IV. Thus the rank of $\mathcal{Q}_{\Delta I}$ is at most $2k+1$. Assumption $P_0 \in \mathcal{P}$ and Eq. (29) or (62) can be used to recover $\mathcal{Q}_0$ and $\boldsymbol{\pi}_0$ once after $\mathcal{Q}_{\Delta I}^0$ is identified.

We now show that Eq. (28) identifies the c.f. $\varphi_{\Delta I}$ (and hence $\mathcal{Q}_{\Delta I}^0$). Recall that Eq. (28) implies that

$$H(u_1,u_2) = \frac{\varphi_{\Delta I}\left(\frac{s_0}{2}u_1, \frac{s_0}{2}u_2\right)}{\varphi_{\Delta I}\left(\frac{s_0}{2}u_1, 0\right)\varphi_{\Delta I}\left(0, \frac{s_0}{2}u_2\right)} \quad \text{for all } (u_1,u_2) \in \overline{\mathcal{V}}^2.$$

Let $\psi_{\Delta I}$ denote a c.f. associated with a candidate $P \in \mathcal{P}$. If the pair $(\tilde{s}, \psi_{\Delta I}(\cdot,\cdot))$ also satisfies Eq. (28), i.e., for all $(u_1,u_2) \in \overline{\mathcal{V}}^2$,

$$\begin{aligned} H(u_1,u_2) &= \frac{\varphi_{\Delta I}\left(\frac{s_0}{2}u_1, \frac{s_0}{2}u_2\right)}{\varphi_{\Delta I}\left(\frac{s_0}{2}u_1, 0\right)\varphi_{\Delta I}\left(0, \frac{s_0}{2}u_2\right)} \\ &= \frac{\psi_{\Delta I}\left(\frac{\tilde{s}}{2}u_1, \frac{\tilde{s}}{2}u_2\right)}{\psi_{\Delta I}\left(\frac{\tilde{s}}{2}u_1, 0\right)\psi_{\Delta I}\left(0, \frac{\tilde{s}}{2}u_2\right)}. \end{aligned} \quad (63)$$

Then on $\overline{\mathcal{V}}^2$, which contains a small neighbourhood of $(0,0)$

$$\begin{aligned} &\varphi_{\Delta I}\left(\frac{s_0}{2}u_1, \frac{s_0}{2}u_2\right)\psi_{\Delta I}\left(\frac{\tilde{s}}{2}u_1, 0\right)\psi_{\Delta I}\left(0, \frac{\tilde{s}}{2}u_2\right) \\ &= \psi_{\Delta I}\left(\frac{\tilde{s}}{2}u_1, \frac{\tilde{s}}{2}u_2\right)\varphi_{\Delta I}\left(\frac{s_0}{2}u_1, 0\right)\varphi_{\Delta I}\left(0, \frac{s_0}{2}u_2\right). \end{aligned} \quad (64)$$

$$\begin{bmatrix}
\pi_k q_{k,-k} & 0 & 0 & 0 & \cdots & \cdots & 0 \\
\pi_{k-1} q_{k-1,-k} & \pi_k q_{k,-k+1} & 0 & 0 & \cdots & \cdots & 0 \\
\pi_{k-2} q_{k-2,-k} & \pi_{k-1} q_{k-1,-k+1} & \pi_k q_{k,-k+2} & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\pi_{-k+2} q_{-k+2,-k} & \pi_{-k+3} q_{-k+3,-k+1} & \pi_{-k+4} q_{-k+4,-k+2} & \cdots & \pi_k q_{k,k-2} & 0 & 0 \\
\pi_{-k+1} q_{-k+1,-k} & \pi_{-k+2} q_{-k+2,-k+1} & \pi_{-k+3} q_{-k+3,-k+2} & \cdots & \pi_{k-1} q_{k-1,k-2} & \pi_k q_{k,k-1} & 0 \\
\pi_{-k} q_{-k,-k} & \pi_{-k+1} q_{-k+1,-k+1} & \pi_{-k+2} q_{-k+2,-k+2} & \cdots & \pi_{k-2} q_{k-2,k-2} & \pi_{k-1} q_{k-1,k-1} & \pi_k q_{k,k} \\
0 & \pi_{-k} q_{-k,-k+1} & \pi_{-k+1} q_{-k+1,-k+2} & \cdots & \pi_{k-3} q_{k-3,k-2} & \pi_{k-2} q_{k-2,k-1} & \pi_{k-1} q_{k-1,k} \\
0 & 0 & \pi_{-k} q_{-k,-k+2} & \cdots & \pi_{k-4} q_{k-4,k-2} & \pi_{k-3} q_{k-3,k-1} & \pi_{k-2} q_{k-2,k} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \pi_{-k} q_{-k,k-2} & \pi_{-k+1} q_{-k+1,k-1} & \pi_{-k+2} q_{-k+2,k} \\
0 & \cdots & 0 & 0 & 0 & \pi_{-k} q_{-k,k-1} & \pi_{-k+1} q_{-k+1,k} \\
0 & \cdots & 0 & 0 & 0 & 0 & \pi_{-k} q_{-k,k}
\end{bmatrix},$$

Box III.

$$\begin{bmatrix}
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & Q_{-k,\circ} \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & Q_{-k+1,\circ} & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & Q_{-k+2,\circ} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 0 & Q_{-1,\circ} & \cdots & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & Q_{0,\circ} & 0 & \cdots & 0 & 0 \\
0 & \cdots & \cdots & 0 & Q_{1,\circ} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & Q_{k-2,\circ} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & Q_{k-1,\circ} & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
Q_{k,\circ} & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0
\end{bmatrix}.$$

Box IV.

Since $\{I_t\}$ is discrete with support $\{-k, \ldots, 0, \ldots, +k\}$, $\varphi_{\Delta I}(\cdot,\cdot)$ and $\psi_{\Delta I}(\cdot,\cdot)$ are entire c.f. Therefore $\varphi_{\Delta I}(\cdot,\cdot)$ and $\psi_{\Delta I}(\cdot,\cdot)$ have analytic continuations for all complex numbers $(z_1, z_2) \in \mathbb{C}^2$. Furthermore, the analytic continuations, $\varphi_{\Delta I}(z_1, z_2)$ and $\psi_{\Delta I}(z_1, z_2)$ are entire functions, and Eq. (64) is satisfied for all $(z_1, z_2) \in \mathbb{C}^2$.

Let $\mathbb{Z} := \left\{z \in \mathbb{C} : \varphi_{\Delta I}\left(\frac{s_0}{2}z, 0\right) = 0\right\}$, and $\widetilde{\mathbb{Z}} := \left\{z \in \mathbb{C} : \psi_{\Delta I}\left(\frac{\tilde{s}}{2}z, 0\right) = 0\right\}$. In the following we shall show that $\mathbb{Z} = \widetilde{\mathbb{Z}}$. Fix $z_1 = d + fi \in \mathbb{Z}$, where $d, f \in \mathbb{R}$. Then, for any $z \in \mathbb{C}$,

$$\varphi_{\Delta I}\left(\frac{s_0}{2}z_1, \frac{s_0}{2}z\right)\psi_{\Delta I}\left(\frac{\tilde{s}}{2}z_1, 0\right)\psi_{\Delta I}\left(0, \frac{\tilde{s}}{2}z\right) = 0. \tag{65}$$

Define $a(z) = (\exp\left[iz\frac{s_0}{2}(-2k)\right], \ldots, \exp\left[iz\frac{s_0}{2}(-1)\right], 1, \exp\left[iz\frac{s_0}{2}(1)\right], \ldots, \exp\left[iz\frac{s_0}{2}(2k)\right])^\intercal$, then

$$\varphi_{\Delta I}\left(\frac{s_0}{2}z_1, \frac{s_0}{2}z\right) = a(z)^\intercal \mathcal{Q}^0_{\Delta I} a(z_1) = a(z)^\intercal A_{\mathcal{Q}_0,\pi_0} B_{\mathcal{Q}_0} a(z_1).$$

Thus $z \to \varphi_{\Delta I}\left(\frac{s_0}{2}z_1, \frac{s_0}{2}z\right)$ is the null function if and only if $A_{\mathcal{Q}_0,\pi_0} B_{\mathcal{Q}_0} a(z_1) = 0$. Since $A_{\mathcal{Q}_0,\pi_0}$ is of full column rank, $A_{\mathcal{Q}_0,\pi_0} B_{\mathcal{Q}_0} a(z_1) = 0$ if and only if $B_{\mathcal{Q}_0} a(z_1) = 0$. Note the equation given in Box V.

The real part of the first component of $B_{\mathcal{Q}_0} a(z_1)$ equals

$$\begin{aligned}
& q^0_{-k,-k} + q^0_{-k,-k+1}\exp\left(-f\frac{s_0}{2}\right)\cos\frac{ds_0}{2} \\
& \quad + q^0_{-k,-k+2}\exp\left(-2f\frac{s_0}{2}\right)\cos\frac{2ds_0}{2} + \cdots \\
& \quad + q^0_{-k,k}\exp\left(-2kf\frac{s_0}{2}\right)\cos\frac{2kds_0}{2},
\end{aligned} \tag{66}$$

while the real part of the last component of $B_{\mathcal{Q}_0} a(z_1)$ equals

$$\begin{aligned}
& q^0_{k,-k}\exp\left(2kf\frac{s_0}{2}\right)\cos\frac{2kds_0}{2} \\
& \quad + q^0_{k,-k+1}\exp\left((2k-1)f\frac{s_0}{2}\right)\cos\frac{(2k-1)ds_0}{2} + \cdots \\
& \quad + q^0_{k,k-1}\exp\left(f\frac{s_0}{2}\right)\cos\frac{ds_0}{2} + q^0_{k,k}.
\end{aligned} \tag{67}$$

Since $q^0_{k,k} > 1/2$ and $q^0_{-k,-k} > 1/2$, either Eq. (66) or (67) is strictly larger than zero, no matter what value z_1 takes. Therefore, $A_{\mathcal{Q}_0,\pi_0} B_{\mathcal{Q}_0} a(z_1) \neq 0$ and $z \to \varphi_{\Delta I}\left(\frac{s_0}{2}z_1, \frac{s_0}{2}z\right)$ is not the null function. Thus, it is possible to choose $z_2 \in \mathbb{C}$ such that $\varphi_{\Delta I}\left(\frac{s_0}{2}z_1, \frac{s_0}{2}z_2\right) \neq 0$, and $\psi_{\Delta I}\left(0, \frac{\tilde{s}}{2}z_2\right) \neq 0$. Then Eq. (65) leads to $\psi_{\Delta I}\left(\frac{\tilde{s}}{2}z_1, 0\right) = 0$, therefore $\mathbb{Z} \subset \widetilde{\mathbb{Z}}$. A similar argument under the full column rank of $A_{\mathcal{Q},\pi}$ shows that $\widetilde{\mathbb{Z}} \subset \mathbb{Z}$. Therefore $\widetilde{\mathbb{Z}} = \mathbb{Z}$.

Since $\varphi_{\Delta I}\left(\frac{s_0}{2}z, 0\right)$ and $\psi_{\Delta I}\left(\frac{\tilde{s}}{2}z, 0\right)$ have growth order 1, using Hadamard's factorization theorem (see, e.g., Stein and Shakarchi (2003), page 147, Theorem 5.1), we can get that there exists a polynomial R of degree ≤ 1 such that for all $z \in \mathbb{C}$,

$$\varphi_{\Delta I}\left(\frac{s_0}{2}z, 0\right) = \exp(R(z))\psi_{\Delta I}\left(\frac{\tilde{s}}{2}z, 0\right).$$

Since $\varphi_{\Delta I}(0,0) = \psi_{\Delta I}(0,0) = 1$, there exists a complex number c such that $\varphi_{\Delta I}\left(\frac{s_0}{2}z, 0\right) = \exp(cz)\psi_{\Delta I}\left(\frac{\tilde{s}}{2}z, 0\right)$. Furthermore, for all $z \in \mathbb{R}$, $\varphi_{\Delta I}\left(-\frac{s_0}{2}z, 0\right) = \overline{\varphi_{\Delta I}\left(\frac{s_0}{2}z, 0\right)}$ and $\psi_{\Delta I}\left(-\frac{\tilde{s}}{2}z, 0\right) = \overline{\psi_{\Delta I}\left(\frac{\tilde{s}}{2}z, 0\right)}$. It is straightforward to show $c = if$, for some $f \in \mathbb{R}$. According to the support information, the only possible value of f is zero. Therefore, $\varphi_{\Delta I}\left(\frac{s_0}{2}z, 0\right) = \psi_{\Delta I}\left(\frac{\tilde{s}}{2}z, 0\right)$, for all $z \in \mathbb{C}$.

$$B_{\mathcal{Q}_0}a(z_1)=\begin{bmatrix} q^0_{-k,-k}+q^0_{-k,-k+1}\exp\left(iz_1\frac{s_0}{2}\right)+q^0_{-k,-k+2}\exp\left(2iz_1\frac{s_0}{2}\right)+\cdots+q^0_{-k,k}\exp\left(2kiz_1\frac{s_0}{2}\right)\\ q^0_{-k+1,-k}\exp\left(-iz_1\frac{s_0}{2}\right)+q^0_{-k+1,-k+1}+q^0_{-k+1,-k+2}\exp\left(iz_1\frac{s_0}{2}\right)+\cdots+q^0_{-k+1,k}\exp\left((2k-1)iz_1\frac{s_0}{2}\right)\\ q^0_{-k+2,-k}\exp\left(-2iz_1\frac{s_0}{2}\right)+q^0_{-k+2,-k+1}\exp\left(-iz_1\frac{s_0}{2}\right)+q^0_{-k+2,-k+2}+\cdots+q^0_{-k+2,k}\exp\left((2k-2)iz_1\frac{s_0}{2}\right)\\ \vdots\quad\vdots\quad\vdots\quad\vdots\\ q^0_{k-2,-k}\exp\left((-2k+2)iz_1\frac{s_0}{2}\right)+\cdots+q^0_{k-2,k-2}+q^0_{k-2,k-1}\exp\left(iz_1\frac{s_0}{2}\right)+q^0_{k-2,k}\exp\left(2iz_1\frac{s_0}{2}\right)\\ q^0_{k-1,-k}\exp\left((-2k+1)iz_1\frac{s_0}{2}\right)+q^0_{k-1,-k+1}\exp\left((-2k+2)iz_1\frac{s_0}{2}\right)+\cdots+q^0_{k-1,k-1}+q^0_{k-1,k}\exp\left(iz_1\frac{s_0}{2}\right)\\ q^0_{k,-k}\exp\left(-2kiz_1\frac{s_0}{2}\right)+q^0_{k,-k+1}\exp\left((-2k+1)iz_1\frac{s_0}{2}\right)+q^0_{k,-k+2}\exp\left((-2k+2)iz_1\frac{s_0}{2}\right)+\cdots+q^0_{k,k} \end{bmatrix}.$$

Box V.

Since $\varphi_{\Delta I}\left(\frac{s_0}{2}z,0\right)=\varphi_{\Delta I}\left(0,\frac{s_0}{2}z\right)$ and $\psi_{\Delta I}\left(\frac{\widetilde{s}}{2}z,0\right)=\psi_{\Delta I}\left(0,\frac{\widetilde{s}}{2}z\right)$ (by strict stationarity), Eq. (64) leads to

$$\varphi_{\Delta I}\left(\frac{s_0}{2}z_1,\frac{s_0}{2}z_2\right)=\psi_{\Delta I}\left(\frac{\widetilde{s}}{2}z_1,\frac{\widetilde{s}}{2}z_2\right)\quad\text{for all }(z_1,z_2)\in\mathbb{C}^2.$$

Namely, the joint distribution of $\left[\frac{s_0}{2}(I_{t-1}-I_{t-2}),\frac{s_0}{2}(I_t-I_{t-1})\right]$ is identified by Eq. (28). According to the joint support information of $(I_{t-1}-I_{t-2},I_t-I_{t-1})$, $s_0\in\mathcal{S}$ can be identified. Therefore, $(s_0,\varphi_{\Delta I}(\cdot,\cdot))$ is identified.

Eq. (27) implies that for all $u\in\mathbb{R}$,

$$\varphi_{\Delta p,1}(u)=\varphi_\varepsilon(u)\mathbb{E}\left(\exp\left[iu\frac{s_0}{2}(I_t-I_{t-1})\right]\right).$$

Then $\varphi_\varepsilon(u)=\varphi_{\Delta p,1}(u)[\varphi_{\Delta I}\left(\frac{s_0}{2}u,0\right)]^{-1}$ for all $u\in\overline{\mathcal{V}}$. □

In general, $(s_0,\varphi_{\Delta I}(\cdot,\cdot))$ cannot be identified without information about the support, as illustrated by the following example.

Example A.1. $\{I_t\}$ could possibly take values in $\{-2,-1,0,1,2\}$. The true marginal distribution satisfies $\Pr(I_t=-1)=\Pr(I_t=1)=1/2$, and the transition matrix is $[1/3\ 2/3;\ 2/3\ 1/3]$. Define $W_t=1/2\,[I_t-I_{t-1}+e_t]$, with $\{e_t\}$ being independent of $\{I_t\}$, and $\Pr(e_t=-2)=b$, $\Pr(e_t=2)=1-b$.

It is easy to show the joint support of (W_{t-1},W_t) is a subset of equation given in Box II for $k=2$. Therefore, Eq. (28) cannot distinguish $(s,\varphi_{\Delta I}(\cdot,\cdot))$ from $(2s,\varphi_W(\cdot,\cdot))$, where $\varphi_W(\cdot,\cdot)$ is the joint c.f.of (W_{t-1},W_t). Simple calculations show $\Pr(W_{t-1}=-2,W_t=-1)=\Pr(W_{t-1}=-1,W_t=-2)=\frac{1}{9}b^2>0$, $\Pr(W_{t-1}=1,W_t=2)=\Pr(W_{t-1}=2,W_t=1)=\frac{1}{9}(1-b)^2>0$. If one has additional information that $\Pr(I_t=-2)=\Pr(I_t=2)=0$, then it is known that $(-2,-1),(-1,-2),(1,2),(2,1)$, are not in equation given in Box II for $k=1$. Thus one is able to distinguish $(s,\varphi_{\Delta I}(\cdot,\cdot))$ from $(2s,\varphi_W(\cdot,\cdot))$. More generally, let $W_t=c\,[I_t-I_{t-1}+e_t]$, where c is any constant and $\{e_t\}$ is independent of $\{I_t\}$. The joint support of (W_{t-1},W_t) is not a subset of equation given in Box II or $k=1$.

Proof of Theorem 6 Part (2). According to Theorem 6 Part (1), s_0 and the joint distribution of $(I_{t-1}-I_{t-2},I_t-I_{t-1})$ can be identified by Eq. (28). For any fixed integer k, $\{I_t\}$ takes values in $\{-k,\ldots,0,\ldots,+k\}$. The probabilities of the first row and the last row of Expression given in Box II satisfy

$$\pi_{0,k}q^0_{k,-k}Q^0_{-k,\circ}=[\Pr(-2k,0),\Pr(-2k,1),\ldots,\Pr(-2k,2k-1),\Pr(-2k,2k)],\tag{68}$$

$$\pi_{0,-k}q^0_{-k,k}Q^0_{k,\circ}=[\Pr(2k,-2k),\Pr(2k,-2k+1),\ldots,\Pr(2k,-1),\Pr(-2k,0)],\tag{69}$$

where $\Pr(-2k,j)$ and $\Pr(2k,-j)$ denote $\Pr(I_{t-1}-I_{t-2}=-2k,I_t-I_{t-1}=j)$ and $\Pr(I_{t-1}-I_{t-2}=2k,I_t-I_{t-1}=-j)$, respectively. The right-hand side of Eqs. (68) and (69) are identified from Theorem 6 Part (1). In order to identify $Q^0_{k,\circ}$ and $Q^0_{-k,\circ}$, $\pi_{0,k}$, $q^0_{k,-k}$, $\pi_{0,-k}$, $q^0_{-k,k}$ need to be positive, that is satisfied under our assumption. By summing up each elements of Eq. (68) and (69), we get $\pi_{0,k}q^0_{k,-k}=\sum_{j=0}^{2k}\Pr(-2k,j)$ and $\pi_{0,-k}q^0_{-k,k}=\sum_{j=0}^{2k}\Pr(2k,-j)$. Therefore, $Q^0_{k,\circ}$ and $Q^0_{-k,\circ}$ can be identified as

$$Q^0_{-k,\circ}=\frac{[\Pr(-2k,0),\Pr(-2k,1),\ldots,\Pr(-2k,2k-1),\Pr(-2k,2k)]}{\sum_{j=0}^{2k}\Pr(-2k,j)},$$

$$Q^0_{k,\circ}=\frac{[\Pr(2k,-2k),\Pr(2k,-2k+1),\ldots,\Pr(2k,-1),\Pr(-2k,0)]}{\sum_{j=0}^{2k}\Pr(2k,-j)}.$$

Consequently $\pi_{0,k}$ and $\pi_{0,-k}$ can be identified as $\pi_{0,k}=\sum_{j=0}^{2k}\Pr(-2k,j)/q^0_{k,-k}$, $\pi_{0,-k}=\sum_{j=0}^{2k}\Pr(2k,-j)/q^0_{-k,k}$. The probabilities of the second row and the second last row of Expression given in Box II satisfy

$$\Pr(-2k+1,-1)=\pi_{0,k}q^0_{k,-k+1}q^0_{-k+1,-k},$$

$$\Pr(-2k+1,2k)=\pi_{0,k-1}q^0_{k-1,-k}q^0_{-k,k},\tag{70}$$

$$\Pr(-2k+1,j)=\pi_{0,k}q^0_{k,-k+1}q^0_{-k+1,-k+j+1}+\pi_{0,k-1}q^0_{k-1,-k}q^0_{-k,-k+j},$$

$$\textit{for } j=0,1,\ldots,2k-1\tag{71}$$

$$\Pr(2k-1,1)=\pi_{0,-k}q^0_{-k,k-1}q^0_{k-1,k},$$

$$\Pr(2k-1,-2k)=\pi_{0,-k+1}q^0_{-k+1,k}q^0_{k,-k},\tag{72}$$

$$\Pr(2k-1,-j)=\pi_{0,-k}q^0_{-k,k-1}q^0_{k-1,k-1-j}+\pi_{0,-k+1}q^0_{-k+1,k}q^0_{k,k-j},$$

$$\textit{for } j=0,1,\ldots,2k-1.\tag{73}$$

Eqs. (70) and (72) can be used to identify $\pi_{0,k-1}q^0_{k-1,-k}$, $\pi_{0,-k+1}q^0_{-k+1,k}$, $q^0_{k-1,k}$ and $q^0_{-k+1,-k}$. Then Eqs. (71) and (73) can be used to identify $q^0_{-k+1,j}$ for $j=-k+1,\ldots,k$ ($q^0_{k,-k+1}>0$ by assumption) and $q^0_{k-1,j}$ for $j=-k,\ldots,k-1$ ($q^0_{-k,k-1}>0$ by assumption), respectively. Consequently, $\pi_{0,k-1}$ and $\pi_{0,-k+1}$ can be identified. Following the same strategy, the probabilities of the third row and the third last row of Expression given in Box II can be used to identify $\pi_{0,k-2}$, $\pi_{0,-k+2}$, $Q^0_{k-2,\circ}$ and $Q^0_{-k+2,\circ}$. Essentially, the same strategy can be applied sequentially to identify π_0 and $\mathcal{Q}_0$. □

A.3. Additional proofs for Section 5

Proof of Theorem 8. Recall that Assumptions 5 and 10 together imply that (41) holds. $H(u,u')$ is complex-valued unless $(2q_0-1)\sin\left(u'\alpha_0\right)\sin\left(u\beta_0\right)\sin\left(u'\alpha_0-u\beta_0\right)=0$. Note that $\alpha_0\neq 0$,

$$\mathrm{Re}(\overline{\mathcal{V}}^2)=\left\{(u,u')\in\overline{\mathcal{V}}^2:\begin{array}{c}E\left[\sin\left((u'\alpha_0-u\beta_0)I_t\right)\right]E\left[\sin\left(u\beta_0I_t\right)\right]E\left[\sin\left(u'\alpha_0I_t\right)\right]\\+E\left[\cos\left((u'\alpha_0-u\beta_0)I_t\right)\right]E\left[\sin\left(u\beta_0I_t\right)\right]E\left[\cos\left(u'\alpha_0I_t\right)\right]\\+E\left[\sin\left((u'\alpha_0-u\beta_0)I_t\right)\right]E\left[\cos\left(u\beta_0I_t\right)\right]E\left[\cos\left(u'\alpha_0I_t\right)\right]\\-E\left[\cos\left((u'\alpha_0-u\beta_0)I_t\right)\right]E\left[\cos\left(u\beta_0I_t\right)\right]E\left[\sin\left(u'\alpha_0I_t\right)\right]=0\end{array}\right\}$$

Box VI.

$\beta_0>0$ and $q_0\in(0,1)$ by assumption,

$$\frac{\partial^2H(0,0)}{\partial u\partial u'}=\left[1-(2q_0-1)^2\right]\alpha_0\beta_0,$$
$$\Rightarrow \beta_0\left[1-(2q_0-1)^2\right]=\frac{\frac{\partial^2H(0,0)}{\partial u\partial u'}}{\alpha_0}. \tag{74}$$

The left-hand side of Eq. (74) is identified from data. Furthermore, we have $\forall u'\in\overline{\mathcal{V}}$:

$$\frac{\partial H(0,u')}{\partial u}=\beta_0\left[1-(2q_0-1)^2\right]\times\frac{\tan(u'\alpha_0)-i(2q_0-1)\left(\tan(u'\alpha_0)\right)^2}{1+(2q_0-1)^2(\tan(u'\alpha_0))^2}. \tag{75}$$

By plugging Eq. (74) into Eq. (75), we obtain

$$\frac{\partial H(0,u')}{\partial u}\Big/\frac{\partial^2H(0,0)}{\partial u\partial u'}=\frac{\tan(u'\alpha_0)-i(2q_0-1)\left(\tan(u'\alpha_0)\right)^2}{\alpha_0\left(1+(2q_0-1)^2(\tan(u'\alpha_0))^2\right)}. \tag{76}$$

For any $\tilde{u}\in(0,\frac{\pi}{2b})\cap\overline{\mathcal{V}}$, from Eq. (76) we have

$$2q_0-1=-\frac{\mathrm{Im}\left(\frac{\partial H(0,\tilde{u})}{\partial u}/\frac{\partial^2H(0,0)}{\partial u\partial u'}\right)}{\tan(\tilde{u}\alpha_0)\,\mathrm{Re}\left(\frac{\partial H(0,\tilde{u})}{\partial u}/\frac{\partial^2H(0,0)}{\partial u\partial u'}\right)}, \tag{77}$$

and

$$\frac{\tan(\tilde{u}\alpha_0)}{\alpha_0}=\frac{\left[\mathrm{Re}\left(\frac{\partial H(0,\tilde{u})}{\partial u}/\frac{\partial^2H(0,0)}{\partial u\partial u'}\right)\right]^2+\left[\mathrm{Im}\left(\frac{\partial H(0,\tilde{u})}{\partial u}/\frac{\partial^2H(0,0)}{\partial u\partial u'}\right)\right]^2}{\mathrm{Re}\left(\frac{\partial H(0,\tilde{u})}{\partial u}/\frac{\partial^2H(0,0)}{\partial u\partial u'}\right)}. \tag{78}$$

The sign of α_0 can be identified by Eq. (36). The first derivative of the left-hand side of Eq. (78) with respective to α_0 is $\frac{(1+\tan^2(\tilde{u}\alpha_0))\tilde{u}\alpha_0-\tan(\tilde{u}\alpha_0)}{\alpha_0^2}$, which is positive (negative), if $\tilde{u}\alpha_0\in(0,\frac{\pi}{2})$ (if $\tilde{u}\alpha_0\in(-\frac{\pi}{2},0)$). Therefore, $0\neq\alpha_0\in\mathcal{B}_1$ can be identified from Eqs. (36) and (78). Consequently $q_0\in(0,1)$ can be identified from Eq. (77) and $0<\beta_0\in\mathcal{B}$ can be identified from Eq. (74). Finally the c.f. $\varphi_\varepsilon(u)$ is identified from (α_0,β_0) and Eq. (34). These arguments complete the proof. □

Proof of Theorem 9. Under Assumptions 7 and 10, Eq. (42) is satisfied by the true parameter value $(\alpha_0,\beta_0,\varphi_{\pi_0})$. Suppose another pair $(\tilde{\alpha},\tilde{\beta},\varphi_{\tilde{\pi}})$ also satisfies Eq. (42) and $\varphi_{\tilde{\pi}}$ denotes the c.f. associated with another probability mass function $\tilde{\pi}$ satisfying Assumption 7. That is, on $\overline{\mathcal{V}}^2$ we have:

$$H(u,u')=\frac{\varphi_{\pi_0}\left(u'\alpha_0-u\beta_0\right)}{\varphi_{\pi_0}\left(-u\beta_0\right)\varphi_{\pi_0}\left(u'\alpha_0\right)}=\frac{\varphi_{\tilde{\pi}}\left(u'\tilde{\alpha}-u\tilde{\beta}\right)}{\varphi_{\tilde{\pi}}\left(-u\tilde{\beta}\right)\varphi_{\tilde{\pi}}\left(u'\tilde{\alpha}\right)},\quad\text{for all }(u,u')\in\overline{\mathcal{V}}^2, \tag{79}$$

and $H(u,u')$ is analytic for all $(u,u')\in\overline{\mathcal{V}}^2$. Let $\mathrm{Re}(\overline{\mathcal{V}}^2)=\left\{(u,u')\in\overline{\mathcal{V}}^2:\mathrm{Im}\left(H(u,u')\right)=0\right\}$, which can be identified from data.

Case 1: $H(u,u')$ is real for all $(u,u')\in\overline{\mathcal{V}}^2$, i.e. $\mathrm{Re}(\overline{\mathcal{V}}^2)=\overline{\mathcal{V}}^2$.

In this case, I_t has a symmetric distribution and $\varphi_{\pi_0}(u)$ is real valued for all $u\in\mathbb{R}$. Denote $\phi_1(u)=\varphi_{\pi_0}(u\alpha_0)$ and $\phi_2(u)=\varphi_{\tilde{\pi}}(u\tilde{\alpha})$. Note that $\forall w\in\overline{\mathcal{V}}$:

$$\frac{\partial H(0,w)}{\partial u_1}=-\frac{\beta_0}{\alpha_0}\left[\frac{\phi_1'(w)}{\phi_1(w)}-\phi_1'(0)\right]=-\frac{\tilde{\beta}}{\tilde{\alpha}}\left[\frac{\phi_2'(w)}{\phi_2(w)}-\phi_2'(0)\right]\Longrightarrow\frac{\phi_1'(w)}{\phi_1(w)}=\frac{\alpha_0}{\beta_0}\frac{\tilde{\beta}}{\tilde{\alpha}}\frac{\phi_2'(w)}{\phi_2(w)}, \tag{80}$$

where $\phi_1'(0)=\phi_2'(0)=0$, since I_t is symmetrically distributed.

Since $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are entire characteristic functions of growth order 1, we have $\forall z\in\mathbb{C}$ (see, e.g., Stein and Shakarchi (2003), page 147, Theorem 5.1):

$$\phi_1(z)=\exp\left(P_1(z)\right)\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}\right),$$
$$\phi_2(z)=\exp\left(P_2(z)\right)\prod_{n=1}^{\infty}\left(1-\frac{z}{b_n}\right)\exp\left(\frac{z}{b_n}\right),$$

where $P_1(z)$ and $P_2(z)$ are polynomials of degree ≤ 1, $\{a_1,a_2,\ldots\}$ and $\{b_1,b_2,\ldots\}$ denote (non-zero) zeros of $\phi_1(\cdot)$ and $\phi_2(\cdot)$, respectively. According to Proposition 3.2. of Stein and Shakarchi (2003) (page 141), we have

$$\frac{\phi_1'(z)}{\phi_1(z)}=a_0+\sum_{n=1}^{\infty}\left(\frac{1}{a_n}+\frac{1}{z-a_n}\right),\quad\forall z\in\mathbb{C}/\{a_1,a_2,\ldots\}$$
$$\frac{\phi_2'(z)}{\phi_2(z)}=b_0+\sum_{n=1}^{\infty}\left(\frac{1}{b_n}+\frac{1}{z-b_n}\right),\quad\forall z\in\mathbb{C}/\{b_1,b_2,\ldots\} \tag{81}$$

where $a_0=P_1'(z)$ and $b_0=P_2'(z)$. Eqs. (80) and (81) imply $\{a_1,a_2,\ldots\}=\{b_1,b_2,\ldots\}$. Therefore, we can get that there exists a polynomial R of degree ≤ 1 such that for all $z\in\mathbb{C}$, $\phi_1(z)=\exp(R(z))\phi_2(z)$. Using the similar argument as in the proof of Theorem 6 Part (1), we can show that for all $z\in\mathbb{C}$,

$$\phi_1(z)=\exp(ifz)\phi_2(z), \tag{82}$$

for some $f\in\mathbb{R}$. Since $\Pr(I_t=-k_1)>0$ and $\Pr(I_t=k_2)>0$, Eq. (82) implies $f=-k_1(\alpha_0-\tilde{\alpha})=k_2(\alpha_0-\tilde{\alpha})$. Therefore, $\alpha_0=\tilde{\alpha}$, $f=0$ and $\phi_1(z)=\phi_2(z)$, $\varphi_{\pi_0}(z)=\varphi_{\tilde{\pi}}(z)$. Along with Eq. (80), we have $\beta_0=\tilde{\beta}$.

Case 2: $\mathrm{Re}(\overline{\mathcal{V}}^2)\subsetneq\overline{\mathcal{V}}^2$.

In this case, I_t is not symmetrically distributed and $\varphi_{\pi_0}(u)$ is complex valued except for some isolated points $\mathrm{Re}(\overline{\mathcal{V}}^2)$ (see equation given in Box VI.) includes some isolated vertical lines, horizontal lines, and straight lines with the same slope $\frac{\beta_0}{\alpha_0}$. Therefore, we can identify $\frac{\beta_0}{\alpha_0}$ from $\mathrm{Re}(\overline{\mathcal{V}}^2)$.

Denote $h(u)=\varphi_{\tilde{\pi}}\left(u\frac{\tilde{\alpha}}{\alpha_0}\right)$, and $u_1=-u\beta_0$, $u_2=u'\alpha_0$. Thus $u'\tilde{\alpha}=u_2\frac{\tilde{\alpha}}{\alpha_0}$, and $-u\tilde{\beta}=u_1\frac{\tilde{\beta}}{\beta_0}\frac{\alpha_0}{\tilde{\alpha}}\frac{\tilde{\alpha}}{\alpha_0}=u_1\frac{\tilde{\alpha}}{\alpha_0}$, because $\frac{\beta_0}{\alpha_0}=\frac{\tilde{\beta}}{\tilde{\alpha}}$. Note that $\varphi_{\pi_0}(\cdot)$, $\varphi_{\tilde{\pi}}(\cdot)$, $h(\cdot)$ are all analytic on $\mathbb{R}$ and equal to 1 at 0. There exists a small neighbourhood $\mathcal{M}$ of $(0,0)\subset\overline{\mathcal{V}}^2$, such that

$\varphi_{\pi_0}(u_1)$, $\varphi_{\pi_0}(u_2)$, $\varphi_{\pi_0}(u_1+u_2)$, $h(u_1)$, $h(u_2)$ and $h(u_1+u_2)$ are all bounded away from zero on $(u_1,u_2)\in\mathcal{M}$. Eq. (79) gives

$$\frac{\varphi_{\pi_0}(u_1+u_2)}{h(u_1+u_2)}=\frac{\varphi_{\pi_0}(u_1)}{h(u_1)}\frac{\varphi_{\pi_0}(u_2)}{h(u_2)}.$$

Then following the similar strategy as in the proof of Theorem 5, we can identify $(\alpha_0,\varphi_{\pi_0})$. Then together with $\mathrm{Re}(\overline{\mathcal{V}}^2)$, we can identify β_0. Finally the c.f. $\varphi_\varepsilon(u)$ is identified from $(\alpha_0,\beta_0,\varphi_{\pi_0})$ and Eq. (34). These arguments complete the proof. □

A.4. Additional proofs for Sections 6 and 7

Proof of Theorem 10. Recall that, under Assumption 12, we have the following Eq. (45):

$$h(u)=\left(\mathbb{E}\left[\cos\left(u\frac{s_t}{2}\right)\right]\right)^2 \quad \text{for all } u\in\overline{\mathcal{V}}.$$

Since $\cos\left(u\frac{a}{2}\right)\geq 0$ for all $u\in\left(-\frac{\pi}{\bar{s}},\frac{\pi}{\bar{s}}\right)\cap\overline{\mathcal{V}}$ and all $a\in[0,\bar{s}]$, we have:

$$0\leq\mathbb{E}\left[\cos\left(u\frac{s_t}{2}\right)\right]=\int_0^{\bar{s}}\cos\left(u\frac{a}{2}\right)dF_s(a)$$
$$=\sqrt{h(u)} \quad \text{for all } u\in\left(-\frac{\pi}{\bar{s}},\frac{\pi}{\bar{s}}\right)\cap\overline{\mathcal{V}}. \tag{83}$$

Let $\varphi_s(\cdot)$ denote the true unknown c.f. of s_t. Since $s_t\in[0,\bar{s}]$ with probability 1 (Assumption 13), $\varphi_s(\cdot)$ is an entire c.f. (see, e.g., Theorem 3.2. of Lukacs (1972)). Eq. (83) can be rewritten as

$$\mathrm{Re}\left(\varphi_s\left(\frac{u}{2}\right)\right)=\frac{1}{2}\varphi_s\left(\frac{u}{2}\right)+\frac{1}{2}\varphi_s\left(\frac{-u}{2}\right)$$
$$=\sqrt{h(u)}, \quad \text{for all } u\in\left(-\frac{\pi}{\bar{s}},\frac{\pi}{\bar{s}}\right)\cap\overline{\mathcal{V}}. \tag{84}$$

Eq. (84) gives the identification of $\mathrm{Re}(\varphi_s(\cdot))$ in $\left(-\frac{\pi}{\bar{s}},\frac{\pi}{\bar{s}}\right)\cap\overline{\mathcal{V}}$ which contains a small neighbourhood of zero. Because $\mathrm{Re}(\varphi_s(\cdot))$ has an analytic continuation for any complex number in the complex plane, we can identify $\mathrm{Re}(\varphi_s(\cdot))$ on the real line.

Define the random variable $W_t=s_tI_t$ and let $G(\cdot)$ be its distribution function. Simple calculation shows $\mathrm{Re}(\varphi_s(\cdot))$ is the c.f. of W_t. Therefore, we can identify $G(\cdot)$. Furthermore, it satisfies :

$$G(w)=\Pr(W_t\leq w)=\frac{1}{2}\Pr(s_t\leq w)+\frac{1}{2}\Pr(-s_t\leq w)$$
$$=\begin{cases}\frac{1}{2}F_s(w)+\frac{1}{2} & w\geq 0\\ \frac{1}{2}-\frac{1}{2}F_s(-w) & w<0.\end{cases} \tag{85}$$

Under Assumption 13, $F_s(w)=0$ for $w\leq 0$, therefore Eq. (85) identifies $F_s(\cdot)$. This completes the proof of the theorem. □

Proof of Theorem 11. Assumption 15 implies that for $j=1,\ldots,n$, $\Pr\left(I_{j,t}=1\right)=\Pr\left(I_{j,t}=-1\right)=\frac{1}{2}$, since $Y_{j,t}^*$ follows a zero mean normal distribution. Under Assumption 15, $q_{jk}\in[0,\frac{1}{2}]$ is strictly increasing in $\omega_{jk}\in[-1,1]$, i.e. g is a strictly increasing function. Furthermore, we have

$$\Pr\left(I_{j,t}=1,I_{k,t}=1\right)=q_{jk},\quad \Pr\left(I_{j,t}=1,I_{k,t}=-1\right)=\frac{1}{2}-q_{jk},$$
$$\Pr\left(I_{j,t}=-1,I_{k,t}=1\right)=\frac{1}{2}-q_{jk}.$$

Let $\widetilde{u}^{jk}=\left(0,\ldots,0,\tilde{u},0,\ldots,0,\tilde{u},0,\ldots,0\right)^\intercal\in\mathbb{R}^n$, where the jth and kth elements of $\widetilde{u}^{jk}$ are equal to $\tilde{u}>0$ and all the other elements are zero. Eqs. (49) and (50) lead to: for $j,k=1,\ldots,n$ and $j\neq k$.

$$H(\widetilde{u}^{jk},\widetilde{u}^{jk})$$
$$=\left[(1-2q_{jk})\cos\frac{\tilde{u}(s_{j,0}-s_{k,0})}{2}+2q_{jk}\cos\frac{\tilde{u}(s_{j,0}+s_{k,0})}{2}\right]^{-2}.$$

Choose a small positive $\tilde{u}$, such that $\cos\frac{\tilde{u}(s_{j,0}-s_{k,0})}{2}>0$, $\cos\frac{\tilde{u}(s_{j,0}+s_{k,0})}{2}>0$, and $\cos\frac{\tilde{u}(s_{j,0}+s_{k,0})}{2}\neq\cos\frac{\tilde{u}(s_{j,0}-s_{k,0})}{2}$. Thus, q_{jk} is uniquely solved as

$$q_{jk}=\frac{[H(\widetilde{u}^{jk},\widetilde{u}^{jk})]^{-1/2}-\cos\frac{\tilde{u}(s_{j,0}-s_{k,0})}{2}}{2\left(\cos\frac{\tilde{u}(s_{j,0}+s_{k,0})}{2}-\cos\frac{\tilde{u}(s_{j,0}-s_{k,0})}{2}\right)}. \tag{86}$$

Thus we obtain the theorem. □

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