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Abstract

We develop the limit theory of the quantilogram and cross-quantilogram under long memory. We establish the sub-root-n central limit theorems for quantilograms that depend on nuisance parameters. We propose a moving block bootstrap (MBB) procedure for inference and we establish its consistency thereby enabling a consistent confidence interval construction for the quantilograms. The newly developed reduction principles for the quantilograms serve as the main technical devices used to derive the asymptotics and establish the validity of MBB. We report some simulation evidence that our methods work satisfactorily. We apply our method to quantile predictive relations between financial returns and long-memory predictors.

Keywords: Long Memory, Moving Block Bootstrap, Nonlinear Dependence, Quantilogram and Cross-Quantilgoram, Reduction Principle.

JEL classification: C22

1 Introduction

Quantile dependence has attracted growing attention in economics, statistics and finance. Unlike the traditional linear dependence statistic, the quantile dependence statistic can capture nonlinear dependence structures at different quantiles. Moreover, the estimation and testing procedures are robust to outliers/heavy tails, which make the statistic well suited to financial applications. Linton and Whang (2007) introduced a time domain statistic called the *quantilogram.* This statistic, which is the correlation between the quantile-hit processes, has been recently extended to a multivariate version called the cross-quantilogram (Han et al., 2016). Hagemann (2011), Li (2008, 2012), and Dette et al. (2015) suggested various frequency domain versions of the quantilogram. See Koenker (2017, Section 4) for a recent review.

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Under some weak dependence assumptions such as ergodicity (Linton and Whang, 2007) or strong mixing (Han et al., 2016), limit theories of the quantilogram and cross-quantilogram were developed. Because of the nuisance parameters in the limit, the statistical inference is typically performed by resampling methods (See Han et al., 2016). The results of frequency domain analysis also relied on the weak dependence assumption, except for a very recent paper by Birr et al. (2017). The last paper considers locally stationary time series.

None of the existing papers study quantile dependence under the presence of a stronger form of dependence such as long-memory. However, in economic and financial applications, we frequently observe series with slowly decaying memory. For example, the squared returns or the (logarithm of) financial volatility series typically show a non-negligible autocorrelation function even with a very long lag. There is thus a large literature on long memory modeling in economics and finance; to name a few, Granger (1980), Baillie (1996) and Doukhan et al. (2002). Considering the importance of the risk-return trade-off in economics and the commonly observed long-memory behavior of financial volatility series, there is ample motivation to study the quantile dependence structure of these long-memory sequences.

This paper develops large sample approximations for the quantilogram and the cross-quantilogram when the data processes show long-memory behaviors. We assume a prototypical linear stationary long-memory process and develop the quantilogram limit theory under this framework. The results show that the asymptotic distributions of the quantilogram and cross-quantilogram under long memory are strikingly different from the weakly dependent cases. The convergence rate is affected by the strong memory property, and it is slower than the usual $\sqrt{n}$-rate. More interestingly, a well-established result from probability theory, the *uniform reduction principle* plays a central role in the limit theory development. See Ho and Hsing (1996, 1997) and Koul and Surgailis (2002) for the uniform reduction principle for long-memory processes. The limit theories of the sample quantile, the quantilogram and the cross-quantilogram are shown to follow interesting new versions of *reduction principle (RP, henceforth)*[1]. As a result, they become asymptotically equivalent to a scaled sample mean. Some nonstandard nuisance parameters also appear in these asymptotic distributions. From these surprising results, we conclude that ignoring the presence of strong dependence will lead to a misleading statistical inference for the quantile-based dependence statistics.

We also provide a valid inferential method for quantilograms using the moving block bootstrap (MBB, henceforth). There are so far no MBB consistency results for this type of nonlinear test statistic with long memory data. We prove MBB consistency by deriving the MBB versions of RP for the sample quantile and quantilograms, which we call MBB-RPs. These results are also new and of independent interest. The stochastic order of all these MBB sample statistics are smaller than those of the original test statistics. This is because the dependence structure is weakened from the blockwise-iid MBB sampling. Nonetheless, MBB consistency is still achieved with a corrected rate of convergence, together with the confirmed asymptotic normality in this paper. This result is in line with the existing results for the long-memory mean case (Lahiri, 2003; Kim and Nordman, 2011; see also Tewes (2016) for a related result for the empirical processes). This result validates the use of the MBB percentile method (Efron, 1979) for statistical inference.

The paper is organized as follows. Section 2 introduces the model and assumptions. Section 3 develops the asymptotic theory of the quantilogram and cross-quantilogram under long memory. Section 4 proposes the MBB inferential method, whose validity is established through MBB-RPs. Some Monte Carlo simulation evidence is provided in Section 5, substantiating the theories and the inferential methods developed in this paper. In the last Section 6, we apply the MBB procedure to examine the quantile-to-quantile predictive relations between the financial return premia and a few commonly used long-memory lagged predictors. Most of the technical proofs are relegated to the Appendix, where a brief review of the main probabilistic technique is also provided.

2 Model and Assumptions

We consider the following stationary linear long memory m by 1 vector process y_t :

$$y_t = \mu_y + \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \tag{1}$$

where $m = 1$ for Section (3.1; quantilogram) and $m = 2$ for Section (3.2; cross-quantilogram). For exposition, we do not distinguish notation between the two cases but it should be straightfoward to see which case is under consideration. When $m = 2$, the memory parameters are allowed to be different with each other, see Section 3.2.

We assume that the marginal τ-quantile of y_t, say ξ_τ, which satisfies $\Pr(y_t \leq \xi_\tau) = F(\xi_\tau) = \tau$, is well defined. We denote by F and $F^{(i)}$ (F_ε and $F_\varepsilon^{(i)}$) the distribution function of y_t (ε_t) and its i-th derivatives, respectively. We assume that $\mu_y = 0$ for simplicity, otherwise letting $y_t^\mu = y_t - \mu_y$ will cover all the following theory.

Assumptions

A1. Let $a_0 = 1$ and $a_j = j^{-\beta} L(j)$, $j \geq 1$, for $\beta = 1 - d \in \left(\frac{1}{2}, 1\right)$ with a slowly varying function $L(\cdot)$ [2].

A2. $\varepsilon_i \sim iid\ (0, \sigma_\varepsilon^2)$ with $E\left(\varepsilon_i^4\right) < \infty$, and $\sup_x \left[F_\varepsilon^{(1)}(x) + \left|F_\varepsilon^{(2)}(x)\right|\right] < \infty$.

Assumption A2 implies that $\sup_x \left[F^{(1)}(x) + |F^{(2)}(x)|\right] < \infty$. Assumption 1 is a commonly used condition to model time series long memory. It corresponds to a popular class, the fractionally integrated process $(1-L)^d y_t = \varepsilon_t$ with the same $d = 1 - \beta = H - 1/2$ (with H being the Hurst index). We treat these two specifications equivalently. Hence (1) with d will be denoted as $y_t \sim I(d)$. For $d \in (0, 1/2)$, the process y_t is strictly stationary. The case $d = 0$ is covered by existing work and technically lies outside our framework.

Following Linton and Whang (2007), we define the quantilogram for a stationary process y_t

$$\rho_{\tau k} = \frac{E\left[\psi_\tau(y_t - \xi_\tau)\,\psi_\tau(y_{t-k} - \xi_\tau)\right]}{E\left[\psi_\tau^2(y_t - \xi_\tau)\right]}, \text{ where } \psi_\tau(\cdot) = \tau - \mathbf{1}(\cdot < 0). \tag{2}$$

Given a stationary bivariate time series process (y_{1t}, y_{2t}), the cross-quantilogram is defined as in Han et al. (2016)

$$\rho_{\tau k} = \rho_{(\tau_1,\tau_2),k} = \frac{E\left[\psi_{\tau_1}\left(y_{1,t} - \xi_{\tau_1}\right)\psi_{\tau_2}\left(y_{2,t-k} - \xi_{\tau_2}\right)\right]}{\sqrt{E\left[\psi^2_{\tau_1}\left(y_{1,t} - \xi_{\tau_1}\right)\right]}\sqrt{E\left[\psi^2_{\tau_2}\left(y_{2,t-k} - \xi_{\tau_2}\right)\right]}}. \tag{3}$$

Linton and Whang (2007) calculated numerically $\rho_{\tau k}$ for stationary Gaussian AR(1) processes and showed how it varied with the underlying AR parameter. Using (16) in the Appendix, we show that the decay rate of the quantilogram of *(1)* is the same as that of the correlogram of *(1),* i.e., long memory processes can be identfied from the quantilogram as from the correlogram.

Lemma 2.1 *Suppose that y_t follows the long-memory process (1) under Assumptions 1 and 2. Then the quantilogram $\rho_{\tau k}$ in (2) for each $\tau \in (0,1)$ decays at the same rate as the correlogram ρ_k as $k \to \infty$:*

$$\rho_{\tau k} = O\left(k^{-1+2d}\right).$$

Remark 2.1 *Our regularity conditions are sufficient and may not be necessary. It is known that, with the α-stable laws of ε's, we need $\alpha(1-d) > 2$ to have Gaussian limit theory of the sum of indicator functionals of long-memory series, see Honda (2009) for a concise summary. To focus on the regular cases with the sub-root-n central limit theorem we assume higher moments of the series ε_i than is common in the short memory literature. This is needed for our proof technique, but seems not needed in practice.*

3 Limit Theory

In this section, we develop limit theories for the sample analogues of the quantilogram and the cross-quantilogram under the model and assumptions in Section 2. The RP results for both quantilograms are derived in the Appendix, which are cruicial in delivering the quantilogram limit theory under long memory.

3.1 Quantilogram limit theory

We first define the unscaled sample and the population quantilogram, respectively, as

$$\tilde{\gamma}_{\tau k}(\xi) = \frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau\left(y_t - \xi\right)\psi_\tau\left(y_{t-k} - \xi\right) \text{ and } \gamma_{\tau k}(\xi) = E\left[\psi_\tau\left(y_t - \xi\right)\psi_\tau\left(y_{t-k} - \xi\right)\right].$$

Then the normalized sample quantilogram is defined as:

$$\hat{\rho}_{\tau k} = \frac{\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau\left(y_t-\hat{\xi}_\tau\right)\psi_\tau\left(y_{t-k}-\hat{\xi}_\tau\right)}{\sqrt{\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau^2\left(y_t-\hat{\xi}_\tau\right)}\sqrt{\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau^2\left(y_{t-k}-\hat{\xi}_\tau\right)}} = \frac{\tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right)}{\sqrt{\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau^2\left(y_t-\hat{\xi}_\tau\right)}\sqrt{\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau^2\left(y_{t-k}-\hat{\xi}_\tau\right)}}, \tag{4}$$

where $\hat{\xi}_\tau$ is the estimated sample quantile from (17). It is straightforward to show (see Lemma A.1 in Appendix) by the law of large number for the denominators: $\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau^2\left(y_t-\hat{\xi}_\tau\right) = \tau(1-\tau)+o_p(1)$ and $\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau^2\left(y_{t-k}-\hat{\xi}_\tau\right) = \tau(1-\tau)+o_p(1)$. Hence, the limiting distribution theory of $\hat{\rho}_{\tau k}$ follows from that of $\tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right)$.

To study the asymptotic distribution of $\tilde{\gamma}_{\tau k}(\hat{\xi}_\tau)$, let $\gamma_{\tau k}(\xi_\tau) = E\left[\psi_\tau(y_t-\xi)\psi_\tau(y_{t-k}-\xi)\right]$. For any given $\tau\in(0,1)$, with a mean value $\bar{\xi}$ between $\hat{\xi}_\tau$ and ξ_τ, we have

$$\begin{aligned} n^{\frac{1}{2}-d}\left(\tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right)-\gamma_{\tau k}(\xi_\tau)\right) &= n^{\frac{1}{2}-d}\left(\tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right)-\gamma_{\tau k}\left(\hat{\xi}_\tau\right)\right)+n^{\frac{1}{2}-d}\left(\gamma_{\tau k}\left(\hat{\xi}_\tau\right)-\gamma_{\tau k}(\xi)\right) & (5)\\ &= n^{\frac{1}{2}-d}\left(\tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right)-\gamma_{\tau k}\left(\hat{\xi}_\tau\right)\right)+\left(\left.\frac{\partial\gamma_{\tau k}(\xi)}{\partial\xi}\right|_{\xi=\bar{\xi}}\right)n^{\frac{1}{2}-d}\left(\hat{\xi}_\tau-\xi_\tau\right), \\ &:= \mathbb{A}+\mathbb{B}. & (6)\end{aligned}$$

It is possible to show (see the proof of 7 in Appendix) that

$$\begin{aligned} \mathbb{A} &= n^{\frac{1}{2}-d}\left(\tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right)-\gamma_{\tau k}\left(\hat{\xi}_\tau\right)\right) \\ &= n^{\frac{1}{2}-d}\left(\frac{1}{n}\sum_{t=k+1}^{n}\left\{\psi_\tau(y_t-\xi_\tau)\psi_\tau(y_{t-k}-\xi_\tau)-E\left[\psi_\tau(y_t-\xi_\tau)\psi_\tau(y_{t-k}-\xi_\tau)\right]\right\}\right)+o_p(1) \\ &= \left\{F^{(1)}(\xi_\tau)\right\}^2\frac{\sum_{t=k+1}^{n}\left(y_ty_{t-k}-E\left(y_ty_{t-k}\right)\right)}{n^{\frac{1}{2}+d}}+o_p(1)=o_p(1), & (7)\end{aligned}$$

so the term $\mathbb{A}$ from (6) is degenerate when d¿0. Hence *the limit theory of* $\mathbb{B} = \left(\left.\frac{\partial\gamma_{\tau k}(\xi)}{\partial\xi}\right|_{\xi=\bar{\xi}}\right)n^{\frac{1}{2}-d}\left(\hat{\xi}_\tau-\xi_\tau\right)$ *dominates in (6)* .

Note that $\gamma_{\tau k}(\xi) = E\left[\psi_\tau(y_t-\xi)\psi_\tau(y_{t-k}-\xi)\right] = \Pr\left(y_t<\xi, y_{t-k}<\xi\right)-2\tau F(\xi)+\tau^2$ is continuously differentiable in ξ under Assumption A2. By the continuous mapping theorem,

$$\left(\left.\frac{\partial\gamma_{\tau k}(\xi)}{\partial\xi}\right|_{\xi=\bar{\xi}}\right) = \left(\left.\frac{\partial\gamma_{\tau k}(\xi)}{\partial\xi}\right|_{\xi=\xi_\tau}\right)+o_p(1).$$

Let us define this quantity as:

$$\nabla G_{\tau k} := \left. \frac{\partial \gamma_{\tau k}(\xi)}{\partial \xi} \right|_{\xi=\xi_\tau} = \left. \frac{\partial \Pr(y_t < \xi, y_{t-k} < \xi)}{\partial \xi} \right|_{\xi=\xi_\tau} - 2\tau F^{(1)}(\xi_\tau). \tag{8}$$

Together with Theorem A.2, we have the reduction principle for the quantilogram: the normalized quantilogram can be approximated by a scaled sample mean. See Lemma A.2 in Appendix.

Remark 3.1 *In weakly dependent cases, the first term $\mathbb{A}$ in (5) has the same order of magnitude as the second term $\mathbb{B}$, so both the terms jointly determine the limit theory of the quantilogram. However, in the long-memory case, it has a smaller order so is degenerate. As a result, the joint distribution information between y_t and y_{t-k} only appears through $\nabla G_{\tau k}$ and the quantilogram limit theory follows from the limit theory of the sample mean. We may naturally label this result as reduction principle for quantilograms in this sense. The difference between the correlogram and the quantilogram is therefore even more prominent under long memory.*

Collecting the results from Theorem A.1, Theorem A.2 and Lemma A.1 in the Appendix, we have the following RP:

$$n^{\frac{1}{2}-d}(\hat{\rho}_{\tau k} - \rho_{\tau k}) = \frac{\nabla G_{\tau k}}{\tau(1-\tau)} \left(\frac{1}{n^{\frac{1}{2}+d}} \sum_{t=1}^{n} y_t \right) + o_p(1) = \frac{\nabla G_{\tau k}}{\tau(1-\tau)} Z_d + o_p(1),$$

directly giving us the univariate CLT below.

To define also a multivariate CLT, let $\Delta_n(\tau, k) = n^{\frac{1}{2}-d}(\hat{\rho}_{\tau k} - \rho_{\tau k})$ for any $\tau \in (0,1)$ and $k = 1, 2, \ldots.$ and let $\Delta_n = \{\Delta_n(\tau, k),\ \tau \in \{\tau_1, \ldots, \tau_q\},\ k \in \{k_1, \ldots, k_p\},\ k_j \geq 1, \tau_j \in (0,1)\}$ be the $pq \times 1$ vector containing the $\Delta_n(\tau, k)$ and let H denote the $pq \times 1$ vector containing the corresponding $\nabla G_{\tau k}/\tau(1-\tau)$.

Theorem 3.1 *(CLT for quantilogram under long memory) Suppose that y_t follows the long-memory process (1) under Assumptions 1 and 2. Then, as $n \to \infty$*

$$\Delta_n(\tau, k) \to^d N\left(0, \frac{c_d^2 (\nabla G_{\tau k})^2}{\tau^2 (1-\tau)^2}\right),$$

where $\nabla G_{\tau k}$ is defined in (8) and c_d is given in Theorem A.1. Furthermore, as $n \to \infty$

$$\Delta_n \to^d H \times N\left(0, c_d^2\right).$$

Remark 3.2 *The rate of convergence is slower than $\sqrt{n}$ as expected. The limiting distribution is non-pivotal due to the presence of d, c_d^2 and the density-like nuisance parameter $\nabla G_{\tau k}$. In principle,* *all the nuisance parameters are estimable and we may obtain a confidence interval for $\rho_{\tau k}$ as*

$$C_\alpha = \left\{ \rho : \hat{\rho}_{\tau k} - \frac{z_{\alpha/2}}{\tau(1-\tau)} \sqrt{\frac{\widehat{c}_d^2 \left(\widehat{\nabla G}_{\tau k}\right)^2}{n^{1-2\widehat{d}}}} \leq \rho \leq \hat{\rho}_{\tau k} + \frac{z_{\alpha/2}}{\tau(1-\tau)} \sqrt{\frac{\widehat{c}_d^2 \left(\widehat{\nabla G}_{\tau k}\right)^2}{n^{1-2\widehat{d}}}} \right\}, \tag{9}$$

where hats denote estimated quantities. However, we propose a bootstrap inference method in Section 4 that obviates the need for direct estimation of these difficult-to-estimate quantities. Note that the quantilogram is perfectly correlated across τ and k.

Remark 3.3 *The theory when $d = 0$ (and for a general weakly dependent process) is given in Han et al (2016); it contains two terms in (6) and hence the distribution theory for the quantilogram has a discontinuity at $d = 0$. Mikusheva (2007) investigated uniform inference in autoregressive models between unit root (including $d = 1$ case here) and stationary and local-to-unity alternatives (including $d \in (0, 1/2)$ case here). The local uniform inference on quantilograms between $d = 0$ and $d \in (0, 1/2)$ (or even more generally, $d \in (-1, 1)$) would be interesting but beyond the scope of this paper. There is no available local uniform limit theory in this fractional process framework, even for the linear statistic.*

3.2 Cross-quantilogram limit theory

We next provide the limit theory for the cross-quantilogram where one series is long memory and the other series is short memory. This is motivated from an empirically relevant scenario. Let y_{1t} be the geometric (log) returns with its memory parameter $d_1 = 0$ in the sense that $E\left[\left(\frac{1}{\sqrt{n}} \sum_{t=1}^n y_{1t}\right)^2\right] = O(1)$. Thus y_{1t} can be any process satisfying a root-n central limit theorem. With an abuse of notation, we denote $y_{1t} \sim I(0)$ but this does not mean that we only consider iid process. As illustrated in (12) below, we allow a general long-run dependence structure for y_{1t}. Let y_{2t} be the (log) volatility with $d_2 \in (0, 1/2)$. The risk-return relation is commonly estimated by the predictive regression model

$$y_{1,t+1} = \alpha + \beta y_{2t} + u_t,$$

although the evidence from this "raw" mean regression is very weak, see Bollerslev et al. (2013), for example. Some possible reasons include: (i) the unbalanced nature of the regression, and (ii) the weak mean-to-mean linear relation. The cross-quantilogram can capture a nonlinear predictive relation by providing a complete dependence structure across the quantile-to-quantile relations. For many predictors, there is ample empirical evidence that $y_{2t} \sim I(d_2)$ with $d_2 \in (0, 1/2)$, but stock returns y_{1t} are generally assumed to be $I(0)$. Let $d_2 = d$ for notational simplicity.

We revisit the cross-quantilogram limit theory under this scenario. Define

$$\tilde{\gamma}_{(\tau_1,\tau_2),k}(\xi_1,\xi_2) = \frac{1}{n}\sum_{t=k+1}^{n} \psi_{\tau_1}(y_{1,t}-\xi_1)\,\psi_{\tau_2}(y_{2,t-k}-\xi_2),$$
$$\gamma_{(\tau_1,\tau_2),k}(\xi_1,\xi_2) = E\left[\psi_{\tau_1}(y_{1,t}-\xi_1)\,\psi_{\tau_2}(y_{2,t-k}-\xi_2)\right].$$

Note that $\gamma_{(\tau_1,\tau_2),k}(\xi_1,\xi_2)$ now has two arguments - ξ_1 (evaluated at a marginal τ_1-quantile of y_1) and ξ_2 (evaluated at a marginal τ_2-quantile of y_2). From a similar decomposition as in (5),

$$\begin{aligned}
&\tilde{\gamma}_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right) - \gamma_{(\tau_1,\tau_2),k}\left(\xi_{1,\tau_1},\xi_{2,\tau_2}\right) \qquad (10)\\
=&\tilde{\gamma}_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right) - \gamma_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right) + \gamma_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right) - \gamma_{(\tau_1,\tau_2),k}\left(\xi_{1,\tau_1},\xi_{2,\tau_2}\right)\\
=&\tilde{\gamma}_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right) - \gamma_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right) + \left(\left.\frac{\partial\gamma_{(\tau_1,\tau_2),k}(\xi_1,\xi_2)}{\partial(\xi_1,\xi_2)}\right|_{(\xi_1,\xi_2)=\bar{\xi}}\right)\begin{bmatrix}\hat{\xi}_{1,\tau_1}-\xi_{1,\tau_1}\\ \hat{\xi}_{2,\tau_2}-\xi_{2,\tau_2}\end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}\left(\left.\frac{\partial\gamma_{(\tau_1,\tau_2),k}(\xi_1,\xi_2)}{\partial(\xi_1,\xi_2)}\right|_{(\xi_1,\xi_2)=\bar{\xi}}\right) &= \left(\left.\frac{\partial\gamma_{(\tau_1,\tau_2),k}(\xi_1,\xi_2)}{\partial\xi_1}\right|_{\xi_1=\bar{\xi}_1}, \left.\frac{\partial\gamma_{(\tau_1,\tau_2),k}(\xi_1,\xi_2)}{\partial\xi_2}\right|_{\xi_2=\bar{\xi}_2}\right) \qquad (11)\\
&:= \left(\nabla G_{\tau k,1},\nabla G_{\tau k,2}\right)\end{aligned}$$

following notation from Section 3.1.

Using Theorem A.2,

$$\begin{aligned}
&n^{\frac{1}{2}-d}\left(\hat{\xi}_{2,\tau_2}-\xi_{2,\tau_2}\right)\\
=&\frac{1}{F_2^{(1)}\left(\xi_{2,\tau_2}\right)}\frac{1}{n^{\frac{1}{2}+d}}\sum_{t=1}^{n}\left(\tau_2-\mathbf{1}\left(y_{2t}<\xi_{2,\tau_2}\right)\right)+o_p(1)\\
\to^d& c_d Z \equiv N\left(0,c_d^2\right).
\end{aligned}$$

Since y_1 is a short-memory $I(0)$ variable satisfying the conventional root-n CLT, we have

$$\begin{aligned}
&n^{\frac{1}{2}}\left(\hat{\xi}_{1,\tau_1}-\xi_{1,\tau_1}\right)\\
=&\frac{1}{F_1^{(1)}\left(\xi_{1,\tau_1}\right)}\frac{1}{n^{\frac{1}{2}}}\sum_{t=1}^{n}\left(\tau_1-\mathbf{1}\left(y_{1t}<\xi_{1,\tau_1}\right)\right)+o_p(1)\\
\to^d& cZ \equiv N\left(0,c^2\right),
\end{aligned}$$

where

$$c^2 = F_1^{(1)}\left(\xi_{1,\tau_1}\right)^{-2}\left\{\tau_1(1-\tau_1)+2\sum_{k=1}^{\infty}Cov\left(\mathbf{1}\left(y_{10}<\xi_{1,\tau_1}\right),\mathbf{1}\left(y_{1k}<\xi_{1,\tau_1}\right)\right)\right\}. \qquad (12)$$

From this analysis, we can conclude that the long-memory time series will determine the first order asymptotic theory (with a slower convergence rate). To see this,

$$\begin{aligned}
&\nabla G_{\tau k,1} n^{\frac{1}{2}-d}\left(\hat{\xi}_{1,\tau_1}-\xi_{1,\tau_1}\right)+\nabla G_{\tau k,2} n^{\frac{1}{2}-d}\left(\hat{\xi}_{2,\tau_2}-\xi_{2,\tau_2}\right) \\
=&\nabla G_{\tau k,2} n^{\frac{1}{2}-d}\left(\hat{\xi}_{2,\tau_2}-\xi_{2,\tau_2}\right)+O_p\left(n^{-d}\right) \\
=&\nabla G_{\tau k,2} n^{\frac{1}{2}-d}\left(\hat{\xi}_{2,\tau_2}-\xi_{2,\tau_2}\right)+o_p(1).
\end{aligned}$$

Hence, the longer-memory term $n^{\frac{1}{2}-d}\left(\hat{\xi}_{2,\tau_2}-\xi_{2,\tau_2}\right)$ dominates. Similarly to Section 3.1, we can show the first term in (10) is negligible (See proof of 13 in Appendix),

$$\frac{n}{n^{\frac{1}{2}+d}}\left(\tilde{\gamma}_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right)-\gamma_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right)\right)=o_p(1). \tag{13}$$

Therefore, we have the following RP for the cross-quantilogram:

$$\frac{n}{n^{\frac{1}{2}+d}}\left(\tilde{\gamma}_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right)-\gamma_{(\tau_1,\tau_2),k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right)\right)=\nabla G_{\tau k,2} n^{\frac{1}{2}-d}\left(\hat{\xi}_{2,\tau_2}-\xi_{2,\tau_2}\right)+o_p(1).$$

Collecting these results, we have the CLT for the cross-quantilogram. Similarly to Theorem 3.1, let $\Delta_n((\tau_1,\tau_2),k)=n^{\frac{1}{2}-d}\left(\hat{\rho}_{(\tau_1,\tau_2),k}-\rho_{(\tau_1,\tau_2),k}\right)$ for any $(\tau_1,\tau_2)\in(0,1)\times(0,1)$ and $k=1,2,\ldots$. and let $\Delta_n=\{\Delta_n((\tau_1,\tau_2),k),\ (\tau_1,\tau_2)\in\{\tau_1,\ldots,\tau_q\}\times\{\tau_1,\ldots,\tau_q\},\ k\in\{k_1,\ldots,k_p\},\ k_j\geq 1,\tau_j\in(0,1)\}$ be the $pq\times 1$ vector containing the $\Delta_n((\tau_1,\tau_2),k)$ and let H denote the $pq\times 1$ vector containing the corresponding $\nabla G_{\tau k,2}/\sqrt{\tau_1(1-\tau_1)\tau_2(1-\tau_2)}$.

Theorem 3.2 *(CLT for cross-quantilogram under long memory) Suppose that $y_{1t}\sim I(0)$ and $y_{2t}\sim I(d)$ and that Assumptions 1 and 2 hold. Then,*

$$\Delta_n((\tau_1,\tau_2),k)\to^d N\left(0,\frac{c_d^2\left(\nabla G_{\tau k,2}\right)^2}{\tau_1(1-\tau_1)\tau_2(1-\tau_2)}\right),$$

and

$$\Delta_n\to^d H\times N\left(0,c_d^2\right),$$

where $\nabla G_{\tau k,2}$ is defined in (11), and c_d is given in Theorem A.1 with $y_t=y_{2t}$.

Remark 3.4 *The multivariate distribution theory of Δ_n for mutiple k's in Theorem 3.2 may be used to construct, for instance, the limiting distribution of the variance ratio and Box-Pierce statistics[3].*

4 Statistical Inference

There are several inferential methods with potential validity for the quantilograms under long memory. For example, we may directly estimate the nuisance parameters and use the first order asymptotic normal distributions. Or we may adopt and develop a version of the self-normalization approach, see Shao (2015) for an excellent survey. Alternatively we can use resampling methods, such as the block bootstrap. In this paper, we consider the block bootstrap method in view of its wide applicability and flexibility. Moreover, as mentioned in Remark 3.2, the presence of the density-like nuisance parameter makes the direct estimation method less attractive, in view of the scarce data information at tails.

The moving block bootstrap (Kunsch, 1989; Liu and Singh, 1992) is a resampling method that can accommodate time series data with unknown dependence structure. The main development, however, has been mostly focused on weakly dependent data. In this section, we adopt a version of moving block bootstrap theory into our framework and prove its validity under long memory.

4.1 Block Bootstrap for Quantilogram

We study the moving block bootstrap (MBB) inference method for the quantilogram under long memory. Kim and Nordman (2011) studied the validity of MBB inference for the sample mean of a long memory process similar to (1). Their main idea concerns properly normalizing the bootstrap variance estimator using an inflation factor that is due to long memory, following Lahiri (2003). In spite of the nonlinearity of our quantilogram statistics, a similar strategy can be employed in our framework using the bootstrap version of the RP (MBB-RP) for quantilograms. Through this MBB-RP for the quantile transformation of long-memory process, the MBB quantilogram statistics are essentially the same as those of a scaled version of the MBB mean, so adjusting the variance inflation factor becomes possible.

We now explain the MBB procedure, following standard notation in the literature (e.g., Kreiss and Paparoditis, 2011). Let $\ell < n$ be an integer block length, and let $\mathcal{B}(t) = (y_t,\, y_{t+1},...,y_{t+\ell-1})$ denote a data block with starting point $t \in \{1, ..., n-\ell+1\}$. The block bootstrap is sampling $b = \lfloor n/\ell \rfloor$ blocks *randomly with replacement* from all possible blocks, and concatenating the bootstrapped sample. Resampling overlapping blocks from $\{\mathcal{B}(t) : t = 1, ..., n-\ell+1\}$ yields MBB sample $y_1^*, ..., y_N^*$, of size $N \equiv b\ell$, which is defined as $(\mathcal{B}(I_1), ..., \mathcal{B}(I_b))$. $I_i's$ are *iid* discrete uniform variables on $\{1, ..., n-\ell+1\}$. Let P^*, E^* and var^* denote probability, expectation and variance of the bootstrap distribution conditional on the original sample.

Using the block bootstrap sample, we estimate the MBB sample quantile,

$$\hat{\xi}_\tau^* = \arg\min_{\xi \in \mathbb{R}} \sum_{t=1}^{N} \rho_\tau \left(y_t^* - \xi\right) \tag{14}$$

and compute the MBB quantile autocovariance

$$\tilde{\gamma}_{\tau k}^* \left(\hat{\xi}_\tau^*\right) = \frac{1}{N} \sum_{t=k+1}^{N} \psi_\tau \left(y_t^* - \hat{\xi}_\tau^*\right) \psi_\tau \left(y_{t-k}^* - \hat{\xi}_\tau^*\right). \tag{15}$$

In a similar fashion, the MBB quantilogram $\hat{\rho}^*_{\tau k}$ is defined as

$$\begin{aligned}\hat{\rho}^*_{\tau k} &= \frac{\frac{1}{N}\sum_{t=k+1}^{N}\psi_\tau\left(y_t^* - \hat{\xi}^*_\tau\right)\psi_\tau\left(y_{t-k}^* - \hat{\xi}^*_\tau\right)}{\sqrt{\frac{1}{N}\sum_{t=k+1}^{N}\psi_\tau^2\left(y_t^* - \hat{\xi}^*_\tau\right)}\sqrt{\frac{1}{N}\sum_{t=k+1}^{N}\psi_\tau^2\left(y_{t-k}^* - \hat{\xi}^*_\tau\right)}} \\ &= \frac{\tilde{\gamma}^*_{\tau k}\left(\hat{\xi}^*_\tau\right)}{\sqrt{\frac{1}{N}\sum_{t=k+1}^{N}\psi_\tau^2\left(y_t^* - \hat{\xi}^*_\tau\right)}\sqrt{\frac{1}{N}\sum_{t=k+1}^{N}\psi_\tau^2\left(y_{t-k}^* - \hat{\xi}^*_\tau\right)}}.\end{aligned}$$

Similarly to Section 3, the MBB sample quantile, quantilogram and cross-quantilogram limit theories follow from MBB-RP's. Bootstrapping sample quantiles under weak dependence has been studied (e.g., Sun and Lahiri, 2006). However, MBB consistency for sample quantile as in (14) under long memory is new and of independent interest. Most existing works use subsampling for this type of nonlinear statistics under long memory to accommodate the case of non-central limit theorem (hence bootstrap is likely to fail). In view of the CLT's under long memory established in this paper (Theorem 3.1 and 3.2), we prove the bootstrap consistency of the MBB sample quantile, the quantilogram and the cross-quantilogram in the presence of long memory. For all the Theorems in this Section, we assume that $1/\ell + \ell/n \to 0$ following the existing block bootstrap literature.

In Appendix, we first establish the MBB-RP for the sample quantile under long memory, see Lemma A.4. This is new in the literature to the best of our knowledge. Using Lemma A.4 and the arguments from Section 3.1, we have the MBB-RP for the quantilogram in Lemma A.5, which determines the main limit theory and MBB consistency of the quantilogram.

As a result of the MBB-RP for quantilogram in Lemma A.5, we have the following CLT

$$N^{1/2}\ell^{-d}\left(\hat{\rho}^*_{\tau k} - \hat{\rho}_{\tau k}\right) \to^d N\left(0, \frac{c_d^2\left(\nabla G_{\tau k}\right)^2}{\tau^2\left(1-\tau\right)^2}\right),$$

leading to the MBB consistency of the quantilogram as given below.

Theorem 4.1 (MBB consistency for quantilogram) *Suppose that y_t follows the long-memory process (1) under Assumptions 1 and 2, and that the rate conditions $N = O(n)$ and $\frac{1}{\ell} + \frac{\ell}{n} = o(1)$ hold. Then,*

$$\sup_{x\in\mathbb{R}}\left|P^*\left(N^{1/2}\ell^{-d}\left(\hat{\rho}^*_{\tau k} - \hat{\rho}_{\tau k}\right) \le x\right) - P\left(n^{\frac{1}{2}-d}\left(\hat{\rho}_{\tau k} - \rho_{\tau k}\right) \le x\right)\right| \to_p 0.$$

Remark 4.1 *From Theorem 4.1, the MBB percentile methods by Efron (1979) are valid for confidence interval construction for quantilograms. As a result, we are able to avoid estimating the nuisance parameters in the asymptotic null distributions, except for the memory parameter d. The estimation for d is readily available from the literature (Geweke and Porter-Hudak, 1983; Robinson, 1995; or Shimotsu and Phillips, 2005).*

Remark 4.2 *It should be possible to allow $\ell = \ell(n)$ to be data dependent. For example, we may* *assume $P(L(n) \leq \ell^n \leq U(n)) \to 1$ where $L(n)$ and $U(n)$ are fixed sequences satisfying conditions such as $1 \leq L(n) \leq U(n) \leq n$ and $L(n) \to \infty$ and $U(n) = o(n)$. In practice, however, the block length should be chosen with care, depending on the context of applications. In the simulation section 5, we provide some guidance based on Monte Carlo simulation that is specifically designed to mimic the empirical scenario in Section 6.*

4.2 Block bootstrap for cross-quantilogram

The extension of the results from Section 4.1 to the cross-quantilogram is straightforward. Define the bootstrap version of unscaled cross-quantilogram

$$\tilde{\gamma}^*_{(\tau_1,\tau_2),k}\left(\hat{\xi}^*_{1,\tau_1}, \hat{\xi}^*_{2,\tau_2}\right) = \frac{1}{N}\sum_{t=k+1}^{N} \psi_{\tau_1}\left(y^*_{1,t} - \hat{\xi}^*_{1,\tau_1}\right)\psi_{\tau_2}\left(y^*_{2,t-k} - \hat{\xi}^*_{2,\tau_2}\right).$$

The MBB cross-quantilogram $\hat{\rho}^*_{(\tau_1,\tau_2),k}\left(\hat{\xi}^*_{1,\tau_1}, \hat{\xi}^*_{2,\tau_2}\right)$ is defined analogously. From the above scenario in Section 3.2 (i.e., $y_{2t} \sim I(d)$ with $d \in (0, 1/2)$, but y_{1t} is $I(0)$), the limit theory of y_{2t} (with the stronger memory hence the slower convergence) dominates the asymptotic theory.

Theorem 4.2 (MBB CLT and consistency for cross-quantilogram) *Suppose that y_t follows the long-memory process (1) under Assumptions 1 and 2, and that the rate conditions $N = O(n)$ and $\frac{1}{\ell} + \frac{\ell}{n} = o(1)$ hold. Then,*

$$N^{1/2}\ell^{-d}\left(\hat{\rho}^*_{(\tau_1,\tau_2),k}\left(\hat{\xi}^*_{\tau}\right) - \hat{\rho}_{(\tau_1,\tau_2),k}\right) \to^d N\left(0, \frac{c_d^2\left(\nabla G_{\tau k,2}\right)^2}{\sqrt{\tau_1\left(1-\tau_1\right)\tau_2\left(1-\tau_2\right)}}\right),$$

so that

$$\sup_{x\in\mathbb{R}}\left|P^*\left(N^{1/2}\ell^{-d}\left(\hat{\rho}^*_{(\tau_1,\tau_2),k}\left(\hat{\xi}^*_{\tau}\right) - \hat{\rho}_{(\tau_1,\tau_2),k}\right) \leq x\right) - P\left(n^{\frac{1}{2}-d}\left(\hat{\rho}_{(\tau_1,\tau_2),k} - \rho_{(\tau_1,\tau_2),k}\right) \leq x\right)\right| \to_p 0.$$

To construct a $(1-\alpha)\times 100\%$ bootstrap CI, we use the bootstrap critical values based on Theorem 4.2. Out of 1000 bootstrap replications, we use the $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ empirical quantiles of the simulated values of $N^{1/2}\ell^{-d}\left(\hat{\rho}^*_{(\tau_1,\tau_2),k}\left(\hat{\xi}^*_{\tau}\right) - \hat{\rho}_{(\tau_1,\tau_2),k}\right)$ with the estimated $\hat{d}$. Denote these critical values as $c^*_{\frac{\alpha}{2}}$ and $c^*_{1-\frac{\alpha}{2}}$, respectively.

We now state the asymptotic coverage probability of this bootstrap procedure.

Corollary 4.1 *Suppose the assumptions in Theorem (4.2) hold, then*

$$\lim_{n\to\infty} P\left(c^*_{\frac{\alpha}{2}} \leq n^{\frac{1}{2}-\hat{d}}\left(\hat{\rho}_{(\tau_1,\tau_2),k} - \rho_{(\tau_1,\tau_2),k}\right) \leq c^*_{1-\frac{\alpha}{2}}\right) = 1-\alpha.$$

Remark 4.3 *From Theorem 4.1, $\left[\hat{\rho}_{(\tau_1,\tau_2),k} - n^{\frac{1}{2}-\hat{d}}c^*_{1-\frac{\alpha}{2}},\ \hat{\rho}_{(\tau_1,\tau_2),k} - n^{\frac{1}{2}-\hat{d}}c^*_{\frac{\alpha}{2}}\right]$ is a $(1-\alpha)\times 100\%$ bootstrap confidence interval that can be used in inference for* any value *of $\rho_{(\tau_1,\tau_2),k}$.*

For the hypothesis test of $\mathbb{H}_0 : \rho_{(\tau_1,\tau_2),k} = \rho^0$, we have the correct asymptotic test size from the test statistics under $\mathbb{H}_0$ and the test consistency under $\mathbb{H}_1$. This is a direct consequence of Theorem (4.1), and the proof is given in Appendix.

Corollary 4.2 *(i) Under* $\mathbb{H}_0 : \rho_{\tau k} = \rho^0$, *let* $T_n = n^{\frac{1}{2}-\hat{d}}\left(\hat{\rho}_{(\tau_1,\tau_2),k} - \rho^0\right)$, *then*

$$\lim_{n\to\infty} P\left(T_n > c^*_{\frac{\alpha}{2}} \text{ or } T_n < c^*_{1-\frac{\alpha}{2}}\right) = \alpha.$$

(ii) Under $\mathbb{H}_1 : \rho_{\tau k} \neq \rho^0$,

$$\lim_{n\to\infty} P\left(T_n > c^*_{\frac{\alpha}{2}} \text{ or } T_n < c^*_{1-\frac{\alpha}{2}}\right) = 1.$$

5 Monte Carlo Simulation

In this section we perform a Monte Carlo simulation to evaluate the theory developed in Sections 3 and 4. In particular, we impose a stationary GARCH model structure on y_{1t}, and the long memory structure of (1) on y_{2t}. This simulation design not only accommodates the model structure of Section 4.2 but also the structure of the empirical excercises in Section 6.

5.1 Simulation design

We generate bivariate Gaussian innovations,

$$\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \sim iid\ N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}\right),$$

and construct a stationary GARCH (1,1) returns for y_{1t}:

$$\begin{aligned} y_{1t} &= \begin{cases} 0 & \text{for } t = 0 \\ \sigma_t \varepsilon_{1t} & \text{for } t \geq 1, \end{cases} \\ \sigma_t^2 &= a_0 + a_1 y_{1t-1}^2 + b_1 \sigma_{t-1}^2. \end{aligned}$$

For the parameter values, we impose the estimates[4] from the monthly US equity premium data in Welch and Goyal (2008), which is used in the empirical exercise shown in Section 6.

For y_{2t}, we generate an $I(d)$ process

$$\log \sigma_{2t}^2 = y_{2t} = \mu_2 + (1-L)^{-d_2}\varepsilon_{2t} = \mu_2 + \sum_{j=0}^{\infty} a_{2,j}\varepsilon_{2t-j}$$

where $\mu_2 = 1$, $a_{2j} = j^{-(1-d)}$ with $d \in (0, 1/2)$. In particular, we employ $d \in \{0.2, 0.25, 0.3, 0.35, 0.4, 0.45\}$ to include the long memory predictors in Section 6. As we see from Table 6.1 below, there are several stationary long memory predictors in this range in practice.

Thus, the simulation environment is particularly designed to emulate a financial return series (y_{1t}) and a long memory predictor (y_{2t}), and their quantile-to-quantile predictive relations, as motivated in the introduction. In Section 6 we perform the data analysis in this type of scenario.

We simulate the above scenario, with $k = 1$, hence one lagged relation. It is straightforward to extend the analysis to a larger $k > 1$ but the most common practice in the risk-return relation literature is to use a one-lagged relation.

5.2 Test size results

When $\phi = 0$, ε_{1t} and ε_{2t} are independent at all lags, thereby generating two independent processes of y_{1t} and y_{2t} at all lags. Therefore, we have

$$\rho_{\tau k}\left(y_{1t}, y_{2t-k}\right) = 0$$

for any $k \geq 1$, regardless of the values of d and other parameter values.

After some extensive simulation studies, we found that a block length of $\ell = n^{3/4}$ shows the best size performance in this scenario. This choice satisfies the rate condition $1/\ell + \ell/n = o(1)$ in Section 3, but is larger than the usual choices from the weakly dependent cases. Intuitively, we may need a larger block length to accommodate strong dependence in data.

The empirical rejection frequencies using Corollary 4.2 to test $\mathbb{H}_0 : \rho_{\tau k} = 0$ are presented in Table 5.1 below.

Table 5.1 Empirical Size with $\ell = n^{3/4}$

d	$\tau = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.20	0.084	0.071	0.052	0.059	0.059	0.038	0.059	0.062	0.073
0.25	0.059	0.05	0.048	0.046	0.039	0.059	0.049	0.055	0.055
0.30	0.057	0.038	0.051	0.042	0.033	0.035	0.039	0.047	0.051
0.35	0.057	0.038	0.028	0.031	0.025	0.031	0.036	0.042	0.059
0.40	0.038	0.039	0.034	0.032	0.024	0.024	0.021	0.03	0.044
0.45	0.028	0.022	0.018	0.024	0.022	0.024	0.018	0.022	0.032

The size performance shows satisfactory results, and we provide additional simulation with $\ell = n^{0.7}$ and $\ell = n^{0.8}$ in the Appendix. Note that, this choice of block length is specifically designed to support our empirical applications below. We recommend to use a block length after this kind of simulation exercise that mimic the applications in hands, since there may be different favorable choices of block lengths to other applications.

In sum, the simulation result in this section confirms the validity of the theoretical results in Section 3, in particular, Corollary 4.2, and also supports the empirical work in Section 6.

6 Empirical Illustration: Predicting Equity Quantile-Premium

In this section, we illustrate the usage of our new test developed in Section 4. In particular, we employ the popular data set from Welch and Goyal (2008)[5], and identify the predictive relations between the quantile of risk premium (quantile-premium) and the quantiles of long memory predictors, such as stock variance and inflation.

The equity risk premium is a key quantity in many asset pricing models and risk management for practitioners. The common time series econometric practice is to find some significant lagged predictors for the risk premium of financial returns using the mean regression. There have been recent suggestions, however, to analyze predictive evidence for stock return quantiles away from the median. See, e.g., Fan and Lee (2019), Lee (2016) and Maynard et al. (2011) for the quantile regression examples, and Han et al. (2016) for the cross-quantilogram analysis. In Table 6.1, LW and ELW are the Robinson (1995) and Shimotsu and Phillips (2005) estimators of the long memory parameters (and their confidence intervals), respectively. Please see Section A.5 for the data description, which is from Welch and Goyal (2008).

From Table 6.1, we conclude that the commonly used persistent predictors are typically non-stationary long memory (including unit root case), or stationary long memory. We restrict our attention to selected stationary long memory predictors (*svar* and *infl*), and their quantile-to-quantile predictive relations with the equity premium (*rp-div*).

From Figure 6.1, some meaningful predictive relations are expected. Thus we use the inference procedure using Corollary 4.1 to test $\mathbb{H}_0 : \rho_{(\tau_1,\tau_2),k} = 0$ for $\tau_1 = \tau_2 = \tau \in \{0.1, 0.2, ..., 0.9\}$. All illustration uses the lag of 1 ($k = 1$), hence a one-step ahead quantile predictive relation.

In Table 6.2, we report the estimated cross-quantilograms between (i) equity premium and stock variance, and (ii) equity premium and inflation, and their MBB confidence intervals. As we see from Table 6.2, there are some predictive evidences at a few central quantiles when using inflation (*infl*) as a long memory predictor. The quantile-to-quantile predictive relation appears more prominently in the left or right tails than at the median when using stock variance (*svar*). This may complement the weak risk-return relation from the mean-to-mean analysis and the resulting ”stock return predictability puzzle”. The result is in line with the recent empirical results. Importantly, the newly proposed MBB method enables valid inference for quantile-to-quantile predictive relations in the presence of long memory, thereby enriching the scope of applications of the quantilograms.

7 Conclusion

This paper investigates the quantilogram and cross-quantilogram estimation, limit theory and the statistical inference when the underlying processes exhibit long-range dependence. We show that the rate of convergence is slower than the usual weakly dependent cases. Meanwhile asymptotic normality still holds under a set of reasonable assumptions. The proper normalization is verified in the limit theory, and we construct a valid moving block bootstrap (MBB) inference for testing the null hypothesis that the quantilogram or cross-quantilogram is zero. While developing the theories,

Figure 6.1: Plots of equity premium, stock variance and inflation, 1934:01 - 2017:12

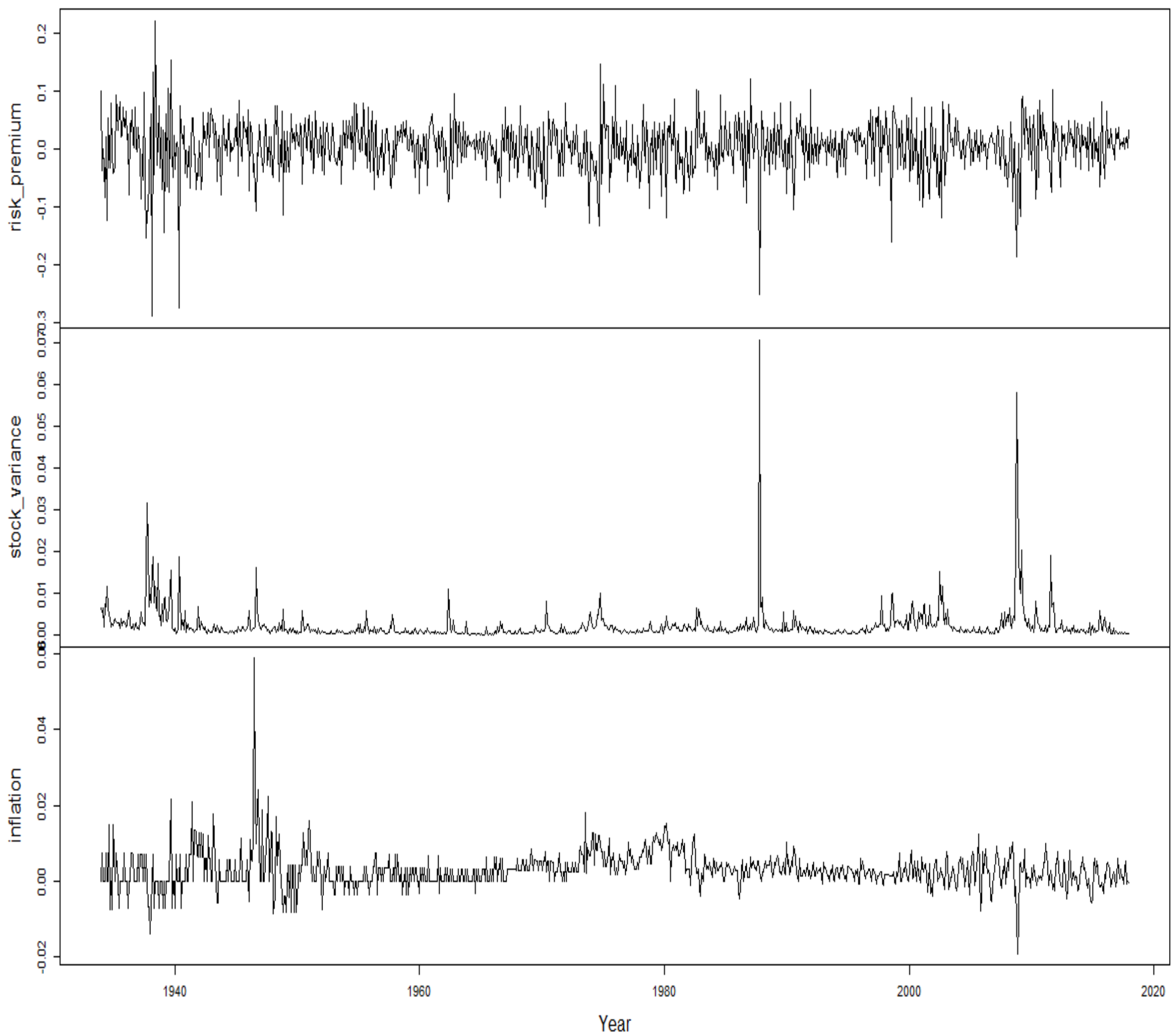

Table 6.1: Memory Parameter Estimations (1934.01-2017.12)

	$\hat{d}$ (LW)	$\hat{d}$ (ELW)
dp	0.93825	0.89495
	(0.89576, 0.98074)	(0.85246, 0.93744)
dy	0.94311	0.88407
	(0.90062, 0.98560)	(0.84158, 0.92656)
ep	0.74790	0.76623
	(0.70541, 0.79039)	(0.72374, 0.80871)
de	0.61011	0.65635
	(0.56762, 0.65260)	(0.61386, 0.69884)
bm	0.87174	0.88072
	(0.82925, 0.91423)	(0.83823, 0.92321)
ntis	0.81681	0.79495
	(0.77432, 0.85930)	(0.75247, 0.83744)
tbl	1.10146	1.10949
	(1.05898, 1.14395)	(1.06700, 1.15197)
tms	0.86697	0.88401
	(0.82448, 0.90946)	(0.84152, 0.92650)
svar	**0.27580**	**0.28475**
	(0.23332, 0.31829)	**(0.24226, 0.32724)**
dfy	0.71884	0.65393
	(0.67635, 0.76133)	(0.61144, 0.69642)
dfr	-0.19570	-0.24445
	(-0.23819, -0.15321)	(-0.28694, -0.20197)
infl	**0.40902**	**0.41393**
	(0.36653, 0.45151)	**(0.37144, 0.45642)**
lty	1.03660	1.0444
	(0.99411, 1.07909)	(1.00198, 1.08696)
ltr	0.09133	0.09341
	(0.04884, 0.13382)	(0.05092, 0.13590)

Table 6.2 Cross-Quantilogram b/w equity premium and stock variance/inflation (1934.01-2017.12)

	Stock variance (svar)			Inflation (infl)		
τ	Estimates	CI_2.5%	CI_97.5%	Estimates	CI_2.5%	CI_97.5%
0.05	**-0.0320***	-0.0547	-0.0061	0.0894	-0.0711	0.1317
0.1	**-0.0445***	-0.1118	-0.0006	0.0096	-0.0759	0.0574
0.2	**-0.0764***	-0.1057	-0.0196	-0.0179	-0.0976	0.0397
0.3	-0.0419	-0.0766	0.0221	**-0.0775***	-0.1346	-0.0114
0.4	-0.0036	-0.0520	0.0650	**-0.1237***	-0.1685	-0.0383
0.5	0.0248	-0.0369	0.0950	**-0.1063***	-0.1352	-0.0480
0.6	0.0584	-0.0177	0.0986	**-0.0699***	-0.1127	-0.0335
0.7	**0.1376***	0.0646	0.1798	**-0.0561***	-0.1091	-0.0189
0.8	**0.1531***	0.0868	0.2006	-0.0392	-0.0754	0.0098
0.9	**0.1323***	0.0503	0.2434	-0.0114	-0.0624	0.0515
0.95	**0.2182***	0.0189	0.2315	-0.0111	-0.0295	0.0423

various new reduction principles (RPs) for the sample quantile and quantilograms are developed. Our simulation results indicate the new MBB quantilogram inference has good size control under some empirically relevant scenarios. The extended data set of Welch and Goyal (2008) is studied to illustrate the benefit of the new inferential methods.

Notes

[1]Our theory focuses on a single quantile level τ, so we label our result as *reduction principle* rather than uniform reduction principle. We appreciate this comment from a referee.

[2]A function $L(\cdot)$ on $[0,\infty)$ is slowly varying at infinity if $L(\cdot)$ is positive on $[A,\infty)$ for some $A>0$, and $\lim_{j\to\infty} L(xj)/L(j)=1$, $\forall x>0$, see Definition 2.3.1 of Giraitis et al. (2012).

[3]See Han et al. (2016) for such a construction under weak dependence.

[4]GARCH(1,1) model estimates $a_0 = 0.000052$, $a_1 = 0.134569$, and $b_1 = 0.854126$, using library(rugarch) in R commands.

[5]We use the monthly data set to 2017, which is available from Amit Goyal's website: http://www.hec.unil.ch/agoyal/

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A Appendix

A.1 Review: expansion of indicator functionals under long memory

We review the results from Ho and Hsing (1996) on the expansion of the indicator functionals of long-memory sequences.

Let $y_t^{(0)} = 1$, and for $p = 1, 2, ...$

$$y_t^{(p)} = \sum_{s_p < ... < s_1 \leq t} a_{t-s_1} ... a_{t-s_p} \varepsilon_{s_1} ... \varepsilon_{s_p}.$$

Then $y_t^{(1)} = y_t$, and $y_t^{(p)}$, for $p = 2, 3, ..,$ will appear in the expansion (16) below. Ho and Hsing (1996) established the following results on the orthogonal processes obtained from (1) under Assumptions A1 and A2:

E1. $y_t^{(p)}$ converges in mean square when $\sum a_i^2 < \infty$ and $E\left[\varepsilon_0^2\right] < \infty$,

E2. $E\left[y_t^{(p)} y_s^{(q)}\right] = 0$ for $p \neq q$, $t \neq s$,

E3. $\left|E\left[y_t^{(p)} y_{t-k}^{(p)}\right]\right| \leq \frac{1}{p!}\left(\sum_{i=0}^{\infty} |a_{t+i} a_i|\right)^p = O\left(k^{-p(1-2d)}\right)$,

E4. $y_t^{(p)}$ is long-memory up to $p(1-2d) < 1$ and short-memory after $p(1-2d) > 1$:

$$E\left[\left(\sum_{t=1}^{n} y_t^{(p)}\right)^2\right] = \begin{cases} O\left(n^{2-p(1-2d)}\right), & p(1-2d) < 1, \\ O(n), & p(1-2d) > 1. \end{cases}$$

Let p^* be the greatest number such that $y_t^{(p^*)}$ is long-memory, i.e., $p^* = \lfloor 1/(1-2d) \rfloor \geq 1$. We have the following expansion of the indicator functional of (1), $\mathbf{1}(y_t < x)$ up to p^*-order:

$$\mathbf{1}(y_t < x) = F(x) - F^{(1)}(x) y_t^{(1)} + F^{(2)}(x) y_t^{(2)} + ... + (-1)^{p^*} F^{(p^*)}(x) y_t^{(p^*)} + R_t(p^*). \tag{16}$$

Let us interpret the orders of magnitudes here as the L_2-norm of the squared sums as given in E4 above. For example, $E\left[\left(\sum_{t=1}^{n} y_t^{(2)}\right)^2\right]$ has a smaller order than $E\left[\left(\sum_{t=1}^{n} y_t^{(1)}\right)^2\right]$ from E4, so we say that $y_t^{(2)}$ is smaller order in probability than $y_t^{(1)}$ and so on, and $R_t(p^*)$ is the smallest remainder (for the detailed expression, see Ho and Hsing (1996)). Since all our proofs will involve either summation or expectation, we can ignore the smaller terms when considering the first order asymptotics. In fact, the major limit theory of our quantilogram asymptotics will follow from the behavior of the first two terms, $F(x) - F^{(1)}(x) y_t^{(1)}$ in (16).

In derving the main results, (16) will play an important role when studying the limit theories of the sample quantiles, quantilogram, cross-quantilogram and the bootstrap versions of them.

A.2 Review: reduction principle (RP) for sample quantile

We now review the available results for the sample quantile limit theory under long memory. The sample quantile is estimated by

$$\hat{\xi}_\tau = \arg\min_{\xi\in\mathbb{R}} \sum_{t=1}^{n} \rho_\tau \left(y_t - \xi\right), \tag{17}$$

where $\rho_\tau \left(x\right) = x \left[\tau - \mathbf{1}\left(x < 0\right)\right]$ is the quantile loss function.

The order of the long-run variance is well known for this case, $E\left[\left(\sum_{t=1}^{n} y_t\right)^2\right] \sim n^{3-2\beta} = n^{1+2d}$, so let

$$\sigma_n = n^{1-(\beta-1/2)} L(n) = n^{1/2+d} L(n).$$

In the sequel, we drop the slowly varying function $L(n)$ and use the normalizer $n^{1/2+d} \sim \sigma_n$ indicating the asymptotic equivalence. The function $L(n)$ may appear in the form of c_d^2 below. However, we propose a bootstrap-based inference, so that ignoring it does not cause a major difference in the theoretical development. Strictly speaking, as gratefully commented by a referee, in the normalizing sequence of the quantilogram the function occurs as $L(n)$, whereas in the bootstrap statistic is $L(\ell)$.

We use the following classical central limit theorem for linear processes (e.g., Ibragimov and Linnik, 1971; Theorem 18.6.5).

Theorem A.1 *(CLT for long-memory linear process) Suppose that y_t follows the long memory process (1) under Assumptions 1 and 2. Then, as $n \to \infty$*

$$\frac{1}{n^{\frac{1}{2}+d}} \sum_{t=1}^{n} y_t \to^d Z_d \equiv N\left(0, c_d^2\right), \quad \textit{where } c_d^2 = \lim_n \mathrm{var}\left(\frac{\sum_{t=1}^{n} y_t}{n^{\frac{1}{2}+d}}\right).$$

Here, c_d^2 is the long-run variance that depends on d.

From Ho and Hsing (1996. See also Beutner et al., 2012; Theorem 2.1), we have the following sample quantile asymptotic normality. Although this result is already established in the literature, we provide a proof using (16), Knight (1998)'s identity and the Convexity Lemma (Pollard, 1991) in Appendix. A similar proof will also carry over to RPs and MBB-RPs for quantilograms in Section 3 and Section 4.

Theorem A.2 *(CLT for sample quantile under long memory) Suppose that y_t follows the long-memory process (1) under Assumptions 1 and 2. Then, the solution of (17) has the following limit theory as $n \to \infty$*

$$n^{\frac{1}{2}-d}\left(\hat{\xi}_\tau - \xi_\tau\right) = \frac{1}{n^{\frac{1}{2}+d}} \sum_{t=1}^{n} y_t + o_p(1) \to^d Z_d \equiv N\left(0, c_d^2\right),$$

where c_d is given in Theorem A.1.

Remark A.1 *In the long-memory case, the limit theory of the sample quantile is the same as that of the sample mean. This is known from the literature, and is one example of the so-called uniform* *reduction rrinciple for the M-estimation under long memory, see Giraitis et al. (2012; Section 10) for the textbook treatment. Dehling and Taqqu (1989) also studied URP for the empirical process under Gaussian subordination. In this paper we will refer to Theorem A.2 as the reduction principle (RP) for sample quantile.*

A.3 Proofs of theorems, supporting lemmas and their proofs

Proof of Lemma 2.1. The proof involves simply applying (16) to the cross-product of indicator functionals, taking expectations and finding the dominating terms for the decay rate using E4. Observe the cross product terms (omitting the remainder terms):

$$
\begin{aligned}
&\{\mathbf{1}(y_t < x) - F(x)\}\{\mathbf{1}(y_{t-k} < x) - F(x)\} \\
=&\left\{F^{(1)}(x)\right\}^2 y_t^{(1)} y_{t-k}^{(1)} - F^{(1)}(x)F^{(2)}(x)y_t^{(1)}y_{t-k}^{(2)} - F^{(1)}(x)F^{(2)}(x)y_t^{(2)}y_{t-k}^{(1)} \\
&+ \left\{F^{(2)}(x)\right\}^2 y_t^{(2)} y_{t-k}^{(2)} + ... + F^{(p^*)}(x)y_t^{(p^*)}y_{t-k}^{(p^*)}.
\end{aligned}
$$

Using the orthogonal property E2 and the order of magnitudes in E3,

$$
\begin{aligned}
&E\left[\{\mathbf{1}(y_t < x) - F(x)\}\{\mathbf{1}(y_{t-k} < x) - F(x)\}\right] \\
=&\left\{F^{(1)}(x)\right\}^2 E\left[y_t^{(1)} y_{t-k}^{(1)}\right] + \left\{F^{(2)}(x)\right\}^2 E\left[y_t^{(2)} y_{t-k}^{(2)}\right] + ... \\
=&O\left(k^{-(1-2d)}\right) + O\left(k^{-2(1-2d)}\right) + ... \\
=&O\left(k^{-(1-2d)}\right),
\end{aligned}
$$

for any given x, proving the claimed result. ■

Proof of Theorem A.2. We combine the standard proof (e.g., Koenker, 2005; Section 4.2) with RP for the indicator functional (16). Define

$$
D_n(\delta) = \frac{1}{n^{2d}} \sum_{t=1}^{n} \left\{\rho_\tau\left(y_t - \xi_\tau - \frac{\delta}{n^{1/2-d}}\right) - \rho_\tau(y_t - \xi_\tau)\right\}
$$

that is convex and is minimized at $n^{\frac{1}{2}-d}\left(\hat{\xi}_\tau - \xi_\tau\right)$.

Using Knight's identity

$$
\rho_\tau(u - v) - \rho_\tau(u) = -v\psi_\tau(u) + \int_0^v [\mathbf{1}(u \le s) - \mathbf{1}(u \le 0)]\, ds,
$$

with $u = y_t - \xi_\tau$ and $v = \frac{\delta}{n^{1/2-d}}$, we have

$$
\begin{aligned}
\rho_\tau\left(y_t - \xi_\tau - \frac{\delta}{n^{1/2-d}}\right) - \rho_\tau(y_t - \xi_\tau) =& -\frac{\delta}{n^{1/2-d}}\psi_\tau(y_t - \xi_\tau) \\
&+ \int_0^{\frac{\delta}{n^{1/2-d}}} [\mathbf{1}(y_t - \xi_\tau \le s) - \mathbf{1}(y_t - \xi_\tau \le 0)]\, ds.
\end{aligned}
$$

Let $D_n(\delta) = D_{n1}(\delta) + D_{n2}(\delta)$, where

$$D_{n1}(\delta) = -\delta\left(\frac{1}{n^{1/2+d}}\sum_{t=1}^{n}\psi_\tau(y_t - \xi_\tau)\right),$$

and

$$D_{n2}(\delta) = \frac{1}{n^{2d}}\sum_{t=1}^{n}\int_0^{\frac{\delta}{n^{1/2-d}}}\left[\mathbf{1}(y_t \le \xi_\tau + s) - \mathbf{1}(y_t \le \xi_\tau)\right]ds.$$

Using (16),

$$\begin{aligned}
\left[\mathbf{1}(y_t \le \xi_\tau + s) - \mathbf{1}(y_t \le \xi_\tau)\right] &= \left(F(\xi_\tau + s) - F^{(1)}(\xi_\tau + s)\,y_t + ...\right) \\
&\quad - \left(F(\xi_\tau) + F^{(1)}(\xi_\tau)\,y_t + ...\right) \\
&= \left(F(\xi_\tau + s) - F(\xi_\tau)\right) \\
&\quad - y_t\left(F^{(1)}(\xi_\tau + s) - F^{(1)}(\xi_\tau)\right) + ... \\
&= sF^{(1)}(\xi_\tau) - sF^{(2)}(\xi_\tau)\,y_t + ...
\end{aligned}$$

so

$$\begin{aligned}
&\int_0^{\frac{\delta}{n^{1/2-d}}}\left[\mathbf{1}(y_t \le \xi_\tau + s) - \mathbf{1}(y_t \le \xi_\tau)\right]ds \\
&= \frac{1}{2}\left(\frac{\delta}{n^{1/2-d}}\right)^2 F^{(1)}(\xi_\tau) - \frac{1}{2}\left(\frac{\delta}{n^{1/2-d}}\right)^2 F^{(2)}(\xi_\tau)\,y_t + ...
\end{aligned}$$

and

$$\begin{aligned}
D_{n2}(\delta) &= \frac{1}{n^{2d}}\frac{1}{2}\left(\frac{\delta}{n^{1/2-d}}\right)^2 F^{(1)}(\xi_\tau)\cdot\sum_{t=1}^{n}1 - \frac{1}{n^{2d}}\frac{1}{2}\left(\frac{\delta}{n^{1/2-d}}\right)^2 F^{(2)}(\xi_\tau)\cdot\sum_{t=1}^{n}y_t + ... \\
&= \frac{\delta^2}{2}F^{(1)}(\xi_\tau) - \frac{\delta^2}{2}F^{(2)}(\xi_\tau)\left(\frac{1}{n}\sum_{t=1}^{n}y_t\right) + ... \\
&= \frac{\delta^2}{2}F^{(1)}(\xi_\tau) + o_p(1),
\end{aligned}$$

since

$$\frac{1}{n}\sum_{t=1}^{n}y_t = \frac{1}{n^{\frac{1}{2}-d}}\left(\frac{1}{n^{\frac{1}{2}+d}}\sum_{t=1}^{n}y_t\right) = O_p\left(\frac{1}{n^{\frac{1}{2}-d}}\right) = o_p(1).$$

Using (16) to $D_{n1}(\delta)$ again,

$$\begin{aligned}
-\delta\left(\frac{1}{n^{1/2+d}}\sum_{t=1}^{n}\psi_\tau\left(y_t-\xi_\tau\right)\right) &= -\delta\left(\frac{1}{n^{1/2+d}}\sum_{t=1}^{n}\left(\tau-\mathbf{1}\left(y_t<\xi_\tau\right)\right)\right)\\
&= -\delta F^{(1)}\left(\xi_\tau\right)\left(\frac{1}{n^{\frac{1}{2}+d}}\sum_{t=1}^{n}y_t\right)+o_p(1)\\
&\to^d -\delta F^{(1)}\left(\xi_\tau\right)Z_d \text{ (from Theorem A.1).}
\end{aligned}$$

Therefore,

$$D_n(\delta)=D_{n1}(\delta)+D_{n2}(\delta)\to^d \frac{\delta^2}{2}F^{(1)}\left(\xi_\tau\right)-\delta F^{(1)}\left(\xi_\tau\right)Z_d := D(\delta).$$

So by the Convexity Lemma (Pollard, 1991; also see Hjort and Pollard, 2011),

$$n^{\frac{1}{2}-d}\left(\hat{\xi}_\tau-\xi_\tau\right)\to^d \arg\min D(\delta)=Z_d.$$

■

Proof of 7. We need to show:

$$\begin{aligned}
&\frac{n}{n^{\frac{1}{2}+d}}\left(\tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right)-\gamma_{\tau k}\left(\hat{\xi}_\tau\right)\right)\\
&=\frac{1}{n^{\frac{1}{2}+d}}\sum_{t=k+1}^{n}\left\{\psi_\tau\left(y_t-\hat{\xi}_\tau\right)\psi_\tau\left(y_{t-k}-\hat{\xi}_\tau\right)-E\left[\psi_\tau\left(y_t-\hat{\xi}_\tau\right)\psi_\tau\left(y_{t-k}-\hat{\xi}_\tau\right)\right]\right\}\\
&=\frac{1}{n^{\frac{1}{2}+d}}\sum_{t=k+1}^{n}\left\{\psi_\tau\left(y_t-\xi_\tau\right)\psi_\tau\left(y_{t-k}-\xi_\tau\right)-E\left[\psi_\tau\left(y_t-\xi_\tau\right)\psi_\tau\left(y_{t-k}-\xi_\tau\right)\right]\right\}+o_p(1).
\end{aligned}$$

Thus, it suffices to show

$$\sup_{\xi_1,\xi_2\in\mathbb{R},\ |\xi_1-\xi_2|<\delta}\left\|\nu_n\left(\xi_1\right)-\nu_n\left(\xi_2\right)\right\|=o_p(1).$$

Using (16),

$$\begin{aligned}
\nu_n(\xi) &= \frac{1}{n^{\frac{1}{2}+d}}\sum_{t=k+1}^{n}\left\{\psi_\tau\left(y_t-\xi\right)\psi_\tau\left(y_{t-k}-\xi\right)-E\left[\psi_\tau\left(y_t-\xi\right)\psi_\tau\left(y_{t-k}-\xi\right)\right]\right\}\\
&= F^{(1)}(\xi)^2\,\frac{\sum_{t=k+1}^{n}\left\{y_t y_{t-k}-E\left[y_t y_{t-k}\right]\right\}}{n^{\frac{1}{2}+d}}+o_p(1).
\end{aligned}$$

Hence, (omitting $\limsup_{n\to\infty}$)

$$
\begin{aligned}
&P\left[\sup_{\xi_1,\xi_2\in\mathbb{R},\ |\xi_1-\xi_2|<\delta}\left\|\nu_n(\xi_1)-\nu_n(\xi_2)\right\|>\eta\right]\\
&=P\left[\sup_{\xi_1,\xi_2\in\mathbb{R},\ |\xi_1-\xi_2|<\delta}\left\|\left(F^{(1)}(\xi_1)^2-F^{(1)}(\xi_2)^2\right)\left(\frac{\sum_{t=k+1}^n\{y_ty_{t-k}-E[y_ty_{t-k}]\}}{n^{\frac{1}{2}+d}}\right)\right\|>\eta\right]\\
&\le P\left[\sup_{\xi_1,\xi_2\in\mathbb{R},\ |\xi_1-\xi_2|<\delta}\left\|\left(\frac{\sum_{t=k+1}^n\{y_ty_{t-k}-E[y_ty_{t-k}]\}}{n^{\frac{1}{2}+d}}\right)\right\|>\frac{\eta}{2\left(\sup F^{(1)}(x)\right)\left|F^{(1)}(\xi_1)-F^{(1)}(\xi_2)\right|}\right],
\end{aligned}
$$

thus

$$
\limsup_{n\to\infty}P\left[\sup_{\xi_1,\xi_2\in\mathbb{R},\ |\xi_1-\xi_2|<\delta}\left\|\nu_n(\xi_1)-\nu_n(\xi_2)\right\|>\eta\right]\to 0,
$$

as long as (i) $\frac{\sum_{t=k+1}^n\{y_ty_{t-k}-E[y_ty_{t-k}]\}}{n^{\frac{1}{2}+d}}=o_p(1)$, and (ii) $F^{(1)}(\cdot)$ is continuous. Note that (ii) is implied by Assumption A2. Therefore, $\nu_n(\cdot)$ is stochastically equicontinuous (around the $n^{\frac{1}{2}-d}$-neighborhood of ξ_τ) if we show (i):

$$
\begin{aligned}
&\left(\frac{1}{n^{\frac{1}{2}+d}}\sum_{t=k+1}^n\left\{\psi_\tau(y_t-\xi_\tau)\psi_\tau(y_{t-k}-\xi_\tau)-E\left[\psi_\tau(y_t-\xi_\tau)\psi_\tau(y_{t-k}-\xi_\tau)\right]\right\}\right)\\
&=\left(F^{(1)}(\xi_\tau)\right)^2\frac{\sum_{t=k+1}^n\left(y_ty_{t-k}-E(y_ty_{t-k})\right)}{n^{\frac{1}{2}+d}}+o_p(1)\\
&=o_p(1).
\end{aligned}
$$

From Theorem 4.5.2 of Giraitis et al. (2012), the product y_ty_{t-k} is short-memory if $d\in(0,1/4)$, so**:**

$$
\frac{1}{\sqrt{n}}\sum_{t=k+1}^n\left(y_ty_{t-k}-E(y_ty_{t-k})\right)=O_p(1),
$$

then

$$
\frac{\sum_{t=k+1}^n\left(y_ty_{t-k}-E(y_ty_{t-k})\right)}{n^{\frac{1}{2}+d}}=\frac{n^{1/2}}{n^{1/2+d}}\frac{1}{\sqrt{n}}\sum_{t=k+1}^n\left(y_ty_{t-k}-E(y_ty_{t-k})\right)=O_p\left(\frac{1}{n^d}\right)=o_p(1).
$$

Furthermore, for $d\in(1/4,1/2)$,

$$
\frac{1}{n^{2d}}\sum_{t=k+1}^n\left(y_ty_{t-k}-E(y_ty_{t-k})\right)=O_p(1),
$$

so from the fact $\frac{1}{2}+d>2d$ for $d\in(1/4,1/2)$,

$$
\frac{\sum_{t=k+1}^n\left(y_ty_{t-k}-E(y_ty_{t-k})\right)}{n^{1/2+d}}=\frac{n^{2d}}{n^{1/2+d}}\left(\frac{1}{n^{2d}}\sum_{t=k+1}^n\left(y_ty_{t-k}-E(y_ty_{t-k})\right)\right)=O_p\left(\frac{n^{2d}}{n^{\frac{1}{2}+d}}\right)=o_p(1).
$$

■

Proof of 13. Since

$$\begin{aligned}
&\psi_{\tau_1}\left(y_{1,t}-\xi_{1,\tau_1}\right)\psi_{\tau_2}\left(y_{2,t-k}-\xi_{2,\tau_2}\right)\\
=&\left(\tau_1-\mathbf{1}\left(y_{1t}<\xi_{1,\tau_1}\right)\right)\left(\tau_2-\mathbf{1}\left(y_{2s}<\xi_{2,\tau_2}\right)\right)\\
=&\left(\tau_1-\mathbf{1}\left(y_{1t}<\xi_{1,\tau_1}\right)\right)\left(F_2^{(1)}(\xi_{2,\tau_2})y_{2,t-k}^{(1)}-F_2^{(2)}(\xi_{2,\tau_2})y_{2,t-k}^{(2)}+...\right),
\end{aligned}$$

and from Lemma 1-7 of Tsay and Chung (2000)

$$\frac{1}{n^{\frac{1}{2}+d}}\sum_{t=1}^{n}\left(\tau_1-\mathbf{1}\left(y_{1t}<\xi_{1,\tau_1}\right)\right)y_{2,t-k}^{(1)}=O_p\left(n^{-d}\right)=o_p(1),$$

and the other terms are smaller orders so negligible. Therefore,

$$\frac{n}{n^{\frac{1}{2}+d}}\left(\tilde{\gamma}_{\tau_1,\tau_2}^{k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right)-\gamma_{\tau_1,\tau_2}^{k}\left(\hat{\xi}_{1,\tau_1},\hat{\xi}_{2,\tau_2}\right)\right)=o_p(1).$$

■

Lemma A.1 $\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau^2\left(y_t-\hat{\xi}_\tau\right)\to^p\tau\left(1-\tau\right)$ *and* $\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau^2\left(y_{t-k}-\hat{\xi}_\tau\right)\to^p\tau\left(1-\tau\right)$ *under (1) with (17).*

Lemma A.2 *(RP for quantilogram) Suppose that y_t follows the long-memory process (1) under Assumptions 1 and 2. Then,*

$$n^{\frac{1}{2}-d}\left(\tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right)-\gamma_{\tau k}\left(\xi_\tau\right)\right)=\left(\nabla G_{\tau k}\right)n^{\frac{1}{2}-d}\left(\hat{\xi}_\tau-\xi_\tau\right)+o_p(1)=\left(\nabla G_{\tau k}\right)\frac{1}{n^{\frac{1}{2}+d}}\sum_{t=1}^{n}y_t+o_p(1)$$

Lemma A.3 *(MBB-RP for indicator functional) For MBB sample $\{y_t^*\}_{t=1}^N$, we will have MBB-RP for indicator functional*

$$\frac{1}{N^{1/2}\ell^d}\sum_{t=1}^{N}\psi_\tau\left(y_t^*-\hat{\xi}_\tau\right)=\left(F^{(1)}(\hat{\xi}_\tau)^*\right)\frac{\sum_{t=1}^{N}y_t^*}{N^{\frac{1}{2}}\ell^d}+o_p(1).$$

Lemma A.4 *(MBB-RP for sample quantile) Suppose that y_t follows the long-memory process (1) under Assumptions 1 and 2. Then,*

$$N^{1/2}\ell^{-d}\left(\hat{\xi}_\tau^*-\hat{\xi}_\tau\right)=\frac{\sum_{t=1}^{N}y_t^*}{N^{\frac{1}{2}}\ell^d}+o_p(1).$$

Lemma A.5 *(MBB-RP for quantilogram) Suppose that (1) and Assumptions 1 and 2 hold. Then,* *as* $n \to \infty$

$$N^{1/2}\ell^{-d}\left(\tilde{\gamma}^*_{\tau k}\left(\hat{\xi}^*_\tau\right)-\tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right)\right)=\nabla G_{\tau k}\left(\frac{\sum_{t=1}^N y^*_t}{N^{\frac{1}{2}}\ell^d}\right)+o_p(1)$$

$$\to^d \quad N(0,(\nabla G_{\tau k})^2\, c_d^2).$$

Proof of Lemma A.1. Note that

$$\begin{aligned}\frac{1}{n}\sum_{t=k+1}^n \psi_\tau^2\left(y_t-\hat{\xi}_\tau\right)&=\frac{1}{n}\sum_{t=k+1}^n\left(\tau-\mathbf{1}\left(y_t<\hat{\xi}_\tau\right)\right)^2\\&=\tau^2-2\tau\frac{1}{n}\sum_{t=k+1}^n\mathbf{1}\left(y_t<\hat{\xi}_\tau\right)+\frac{1}{n}\sum_{t=k+1}^n\mathbf{1}\left(y_t<\hat{\xi}_\tau\right),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n}\sum_{t=k+1}^n\mathbf{1}\left(y_t<\hat{\xi}_\tau\right)&=\frac{F\left(\hat{\xi}_\tau\right)}{n}\sum_{t=k+1}^n 1-\frac{F^{(1)}(\hat{\xi}_\tau)}{n}\sum_{t=k+1}^n y_t\\&=F\left(\hat{\xi}_\tau\right)+o_p(1)=\tau+o_p(1).\end{aligned}$$

So

$$\frac{1}{n}\sum_{t=k+1}^n\psi_\tau^2\left(y_t-\hat{\xi}_\tau\right)=\tau\left(1-\tau\right)+o_p(1).$$

Similarly,

$$\frac{1}{n}\sum_{t=k+1}^n\psi_\tau^2\left(y_{t-k}-\hat{\xi}_\tau\right)=\tau\left(1-\tau\right)+o_p(1).$$

■

Proof of Lemma A.3. We show the following:

$$\frac{1}{N^{1/2}\ell^d}\sum_{t=1}^N\left\{\mathbf{1}\left(y^*_t<\hat{\xi}_\tau\right)-F\left(\hat{\xi}_\tau\right)^*+\left(F^{(1)}(\hat{\xi}_\tau)^*\right)y^*_t\right\}=o_p(1).$$

The proof is using the standard MBB theory, see Tewes (2016), for example. Note that,

$$
\begin{aligned}
&E^*\left[\left(\frac{1}{N^{1/2}\ell^d}\sum_{t=1}^{N}\left\{\mathbf{1}\left(y_t^*<\hat{\xi}_\tau\right)-F\left(\hat{\xi}_\tau\right)^*+\left(F^{(1)}(\hat{\xi}_\tau)^*\right)y_t^*\right\}\right)^2\right]\\
=&\frac{1}{N\ell^{2d}}E^*\left[\left(\sum_{t=1}^{N}\left\{\mathbf{1}\left(y_t^*<\hat{\xi}_\tau\right)-F\left(\hat{\xi}_\tau\right)^*+\left(F^{(1)}(\hat{\xi}_\tau)^*\right)y_t^*\right\}\right)^2\right]\\
=&\frac{1}{N\ell^{2d}}bE^*\left[\left(\sum_{t\in B(i)}\left\{\mathbf{1}\left(y_t^*<\hat{\xi}_\tau\right)-F\left(\hat{\xi}_\tau\right)^*+\left(F^{(1)}(\hat{\xi}_\tau)^*\right)y_t^*\right\}\right)^2\right]\\
=&\frac{1}{N\ell^{2d}}b\frac{1}{n-\ell+1}\sum_{i=1}^{n-\ell+1}\left(\sum_{t\in B(i)}\left\{\mathbf{1}\left(y_t<\hat{\xi}_\tau\right)-F\left(\hat{\xi}_\tau\right)+\left(F^{(1)}(\hat{\xi}_\tau)\right)y_t\right\}\right)^2\\
=&\frac{1}{\ell^{1+2d}}\frac{1}{n-\ell+1}\sum_{i=1}^{n-\ell+1}\left(\sum_{t\in B(i)}\left\{F^{(2)}(\hat{\xi}_\tau)y_t^{(2)}-F^{(3)}(\hat{\xi}_\tau)y_t^{(3)}+...\right\}\right)^2, \text{ using (16).}
\end{aligned}
$$

From the law of iterated expectation, and using the property E2 and E4 above,

$$
\begin{aligned}
&E\left[\left(\frac{1}{N^{1/2}\ell^d}\sum_{t=1}^{N}\left\{\mathbf{1}\left(y_t^*<\hat{\xi}_\tau\right)-F\left(\hat{\xi}_\tau\right)^*+\left(F^{(1)}(\hat{\xi}_\tau)^*\right)y_t^*\right\}\right)^2\right]\\
=&E\left[E^*\left[\left(\frac{1}{N^{1/2}\ell^d}\sum_{t=1}^{N}\left\{\mathbf{1}\left(y_t^*<\hat{\xi}_\tau\right)-F\left(\hat{\xi}_\tau\right)^*+\left(F^{(1)}(\hat{\xi}_\tau)^*\right)y_t^*\right\}\right)^2\right]\right]\\
=&\frac{1}{\ell^{1+2d}}E\left(\sum_{t\in B(i)}\left\{F^{(2)}(\hat{\xi}_\tau)y_t^{(2)}-F^{(3)}(\hat{\xi}_\tau)y_t^{(3)}+...\right\}\right)^2\\
=&O\left(\frac{1}{\ell^{1+2d}}E\left[\left(\sum_{t=1}^{\ell}y_t^{(2)}\right)^2\right]\right)=\left\{\begin{array}{ll} O\left(\frac{\ell^{2d}}{\ell}\right) & \text{, if } d\in(0,1/4)\\ O\left(\frac{1}{\ell^{2d}}\right) & \text{, if } d\in(1/4,1/2)\end{array}\right.\\
=&o(1),
\end{aligned}
$$

giving the required result. ■

Proof of Theorem A.4. Similarly to the proof of Theorem A.2, define

$$
D_n^*(\delta)=\frac{1}{\ell^{2d}}\sum_{t=1}^{N}\left\{\rho_\tau\left(y_t^*-\hat{\xi}_\tau-\frac{\delta}{N^{1/2}\ell^{-d}}\right)-\rho_\tau\left(y_t^*-\hat{\xi}_\tau\right)\right\},
$$

which is convex and minimized at $\delta=N^{1/2}\ell^{-d}\left(\hat{\xi}_\tau^*-\hat{\xi}_\tau\right)$. Note that $\hat{\xi}_\tau$ is fixed conditional on data (under P^*). Therefore, using Knight's identity $D_n^*(\delta) = D_{n1}^*(\delta) + D_{n2}^*(\delta)$:

$$\begin{aligned} D_{n1}^*(\delta) &= -\delta \left(\frac{1}{N^{1/2}\ell^d} \sum_{t=1}^{N} \psi_\tau \left(y_t^* - \hat{\xi}_\tau \right) \right) \\ &= -\delta \left(F^{(1)}(\hat{\xi}_\tau)^* \right) \frac{\sum_{t=1}^{N} y_t^*}{N^{\frac{1}{2}}\ell^d} + o_p(1), \end{aligned}$$

where the last line comes from MBB-RP for indicator functional (Lemma A.3)

$$\begin{aligned} D_{n2}^*(\delta) &= \frac{1}{\ell^{2d}} \sum_{t=1}^{N} \int_0^{\frac{\delta}{N^{1/2}\ell^{-d}}} \left[\mathbf{1}\left(y_t^* < s\right) - \mathbf{1}\left(y_t^* < 0\right) \right] ds \\ &= \frac{1}{\ell^{2d}} \sum_{t=1}^{N} \left(\frac{\delta^2}{2\left(N^{\frac{1}{2}}\ell^{-d}\right)^2} \right) F^{(1)}(\hat{\xi}_\tau)^* + o_p(1) \\ &= \frac{\delta^2}{2} F^{(1)}(\hat{\xi}_\tau)^* + o_p(1). \end{aligned}$$

Thus, by Convexity lemma again

$$\begin{aligned} N^{1/2}\ell^{-d}\left(\hat{\xi}_\tau^* - \hat{\xi}_\tau \right) &= \arg\min D_n^*(\delta) \\ &= \frac{\sum_{t=1}^{N} y_t^*}{N^{\frac{1}{2}}\ell^d} + o_p(1). \end{aligned}$$

■

Proof of Lemma A.5 and Theorem 4.1. To prove Theorem 4.1, it suffices to show

$$\sup_{x\in\mathbb{R}} \left| P^*\left(N^{1/2}\ell^{-d}\left(\tilde{\gamma}_{\tau k}^*\left(\hat{\xi}_\tau^*\right) - \tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right) \right) \le x \right) - \Phi\left(\frac{x}{c_d\left(\nabla G_{\tau k,2}\right)} \right) \right| = o_p(1).$$

From the continuity of $\Phi(\cdot)$ (Van Der Vaart (2000), Lemma 2.11), we only need to show (under P^*, omitted hereafter within this proof)

$$N^{1/2}\ell^{-d}\left(\tilde{\gamma}_{\tau k}^*\left(\hat{\xi}_\tau^*\right) - \tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right) \right) \to^d N\left(0, c_d^2\left(\nabla G_{\tau k,2}\right)^2\right).$$

Note that,

$$\tilde{\gamma}_{\tau k}^*\left(\hat{\xi}_\tau^*\right) - \tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right) = \frac{1}{N}\sum_{t=k+1}^{N} \psi_\tau\left(y_t^* - \hat{\xi}_\tau^*\right)\psi_\tau\left(y_{t-k}^* - \hat{\xi}_\tau^*\right) - \frac{1}{n}\sum_{t=k+1}^{n} \psi_\tau\left(y_t - \hat{\xi}_\tau\right)\psi_\tau\left(y_{t-k} - \hat{\xi}_\tau\right).$$

Following the proof of Proposition B.5 of Han et al. (2016), combined with the proofs of (13), (7), and using Theorem (A.4), we have

$$N^{1/2}\ell^{-d}\left(\tilde{\gamma}_{\tau k}^*\left(\hat{\xi}_\tau^*\right) - \tilde{\gamma}_{\tau k}\left(\hat{\xi}_\tau\right) \right)$$

$$= \nabla G_{\tau k} \left\{ N^{1/2} \ell^{-d} \left(\hat{\xi}_{\tau}^{*} - \hat{\xi}_{\tau} \right) \right\} + o_p(1)$$
$$= \nabla G_{\tau k} \frac{\sum_{t=1}^{N} y_t^{*}}{N^{\frac{1}{2}} \ell^{d}} + o_p(1).$$

Therefore, together with the available MBB-CLT for the mean under long memory (e.g., Theorem 2.1 of Kim and Nordman (2011)):

$$\frac{\sum_{t=1}^{N} y_t^{*}}{N^{\frac{1}{2}} \ell^{d}} \to^{d} N(0, c_d^2),$$

we finally have

$$N^{1/2} \ell^{-d} \left(\tilde{\gamma}_{\tau k}^{*} \left(\hat{\xi}_{\tau}^{*} \right) - \tilde{\gamma}_{\tau k} \left(\hat{\xi}_{\tau} \right) \right) \to^{d} N(0, (\nabla G_{\tau k})^2 \, c_d^2).$$

■

Proof of Corollary 4.2. Corrollary (4.2)-(i) is a direct consequence from Theorem (4.1) under $\mathbb{H}_0 : \rho_{(\tau_1,\tau_2),k} = \rho^0$. Under the alternative hypothesis $\mathbb{H}_1 : \rho_{\tau k} \neq 0$, note that $c^{*}_{\frac{\alpha}{2}}$ and $c^{*}_{1-\frac{\alpha}{2}}$ are bounded in probability from Theorem (4.2), On the other hand, under $\mathbb{H}_1 : \rho_{\tau k} \neq 0$, so for any $d \in (0, 1/2)$, $n^{\frac{1}{2}-\hat{d}} \hat{\rho}_{\tau k} = n^{\frac{1}{2}-d} \left(\rho_{\tau k} + o_p(1) \right) \to \infty$ as $n \to \infty$. Therefore,

$$\lim_{n \to \infty} P \left(n^{\frac{1}{2}-\hat{d}} \hat{\rho}_{\tau k} > c^{*}_{\frac{\alpha}{2}} \text{ or, } n^{\frac{1}{2}-\hat{d}} \hat{\rho}_{\tau k} < c^{*}_{1-\frac{\alpha}{2}} \right) = 1,$$

confirming Corrollary (4.2)-(ii). ■

A.4 Size performances with different block lengths

Table A.1: Empirical Size with $\ell = n^{0.7}$

d	$\tau = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.20	0.058	0.036	0.036	0.03	0.026	0.033	0.024	0.038	0.067
0.25	0.031	0.026	0.023	0.019	0.02	0.039	0.024	0.029	0.048
0.30	0.028	0.015	0.011	0.011	0.021	0.017	0.015	0.017	0.044
0.35	0.023	0.01	0.009	0.01	0.009	0.012	0.012	0.015	0.029
0.40	0.006	0.009	0.01	0.007	0.01	0.012	0.005	0.008	0.022
0.45	0.011	0.005	0.008	0.008	0.004	0.006	0.007	0.011	0.014

Table A.2: Empirical Size with $\ell = n^{0.8}$

d	$\tau = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.20	0.104	0.106	0.1	0.096	0.101	0.115	0.124	0.116	0.125
0.25	0.094	0.092	0.114	0.086	0.096	0.093	0.08	0.096	0.085
0.30	0.119	0.074	0.084	0.073	0.064	0.074	0.089	0.069	0.095
0.35	0.085	0.063	0.067	0.071	0.08	0.078	0.078	0.058	0.097
0.40	0.081	0.072	0.065	0.046	0.074	0.056	0.064	0.07	0.072
0.45	0.08	0.059	0.042	0.064	0.052	0.055	0.051	0.058	0.084

A.5 Data description and plots of equity premium, stock variance and inflation

The variable names (with their abbreviation) follow Welch and Goyal (2008), which we refer for detailed constructions and economic foundations of the data set. The extended data set (up to 2015) is obtain's from Amit Goyal's webpage (http://www.hec.unil.ch/agoyal/)

- "rp_div": Equity Risk Premium (log) (including dividends).
- "dp": Dividend-price ratio (log) - difference between the log of dividends paid on the S&P 500 index and the log of prices, where dividends are measured using a twelve-month moving sum.
- "dy": Dividend yield (log) - difference between the log of dividends and the log of lagged prices.
- "ep": Earnings-price ratio (log) - difference between the log of earnings on the S&P 500 index and the log of prices, where earnings are measured using a twelve-month moving sum.
- "de": Dividend-payout ratio (log) - difference between the log of dividends and log of earnings.
- "svar": Stock variance - sum of squared daily returns on the S&P 500 index. Daily returns for 1871 to 1926 are obtained from Bill Schwert, while daily returns from 1926 to 2005 are obtained from CRSP.
- "bm": Book-to-market ratio - ratio of book value to market value for the Dow Jones Industrial Average.
- "ntis": Net equity expansion - ratio of twelve-month moving sums of net issues by NYSE-listed stocks to total end-of-year market capitalization of NYSE stocks.
- "tbl": T-bill rate - interest rate on a 3-month Treasury bill (secondary market).
- "lty": Long-term yield - long-term government bond yield (Long-term government bond yields for the period 1919 to 1925 is the U.S. Yield On Long-Term United States Bonds series from NBER's Macrohistory database. Yields from 1926 to 2005 are from Ibbotson's Stocks, Bonds, Bills and Inflation Yearbook).

- “ltr”: Long-term return - return on long-term government bonds (Long-term government bond returns for the period 1926 to 2005 are from Ibbotson’s Stocks, Bonds, Bills and Inflation Yearbook).
- “tms”: Term spread - difference between the long-term yield and the T-bill rate.
- “dfy”: Default yield spread - difference between BAA- and AAA-rated corporate bond yields.
- “dfr”: Default return spread - difference between long-term corporate bond and long-term government bond returns.
- “infl”: Inflation - Inflation is the Consumer Price Index (All Urban Consumers) for the period 1919 to 2015 from the Bureau of Labor Statistics. Because inflation information is released only in the following month, in our monthly regressions, we inserted one month of waiting before use. Note since inflation rate data are released in the following month, we use x(i,t-1) for inflation.