Problem Sets for: *The Models and Methods of Financial Econometrics*. Cambridge University Press. January 2019.* Solutions available on request

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1 Predictability

1. Recently, Warren Buffett has predicted that the Dow Jones index will exceed one million in a hundred years time. Given that the current level of the index is 22400, what annual rate of return is he assuming?

2. First obtain daily price data on a stock index and two individual stocks from a market of your choice (some choices below). The calculations can be performed in Excel and/or `Eviews`, but also in other software packages, as you prefer.

   (a) Compute the sample statistics of the stock return (computed from the daily closing price) series, i.e., the mean, standard deviation, skewness and kurtosis. You may ignore dividends and just focus on capital gain.

   (b) The Jarque-Bera statistic is
   $$JB = \frac{n}{6}\left(\widehat{\kappa}_3^2 + \frac{1}{4}\widehat{\kappa}_4^2\right),$$
   where $\widehat{\kappa}_3, \widehat{\kappa}_4$ are the sample skewness and sample excess kurtosis. If the population is i.i.d. with a normal distribution, then JB is asymptotically $\chi^2(2)$. Calculate JB for the data and test the normality hypothesis.

   (c) Compute the first 20 autocorrelation coefficients and test whether the series is linearly predictable or not.

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(d) Does it make a difference whether you compute returns using log price differences or as actual return?

3. The (equal weighted) moving average filter of a series X_t is defined as

$$SMA_t^k = \frac{X_t + X_{t-1} + \ldots + X_{t-k}}{k},$$

where k is the number of lags to include, sometimes called a bandwidth parameter. For daily stock prices, common values include 5, 10, 20, 50, 100, and 200. Compute the SMA for the series. The exponential weighted average is defined as

$$EWMA_t = \alpha X_t + (1-\alpha) EWMA_{t-1},$$

where $EWMA_1 = X_1$ and $\alpha \in (0,1)$. Can relate $\alpha = 2/(k+1)$, where k is number of time periods. These smoothed values are often used in trading strategies of the contrarian type, that is: buy when $X_t < SMA_t^k$ and sell when

4. $X_t > SMA_t^k$, or moment type trading strategies, that is, buy when $X_t > SMA_t^k$ and sell when $X_t < SMA_t^k$. Comment on the efficacy of these trading strategies for your dataset, Faber (2013).

5. The so-called **Bollinger bands** (http://en.wikipedia.org/wiki/Bollinger_Bands) are a modification of the moving average rules that allow a margin of safety by allowing for time varying volatility. They are defined as follows:

$$\begin{aligned} BB_t^U &= SMA_t^k + 2\sigma_t \\ BB_t^L &= SMA_t^k - 2\sigma_t \\ \sigma_t &= std(X_t, X_{t-1}, \ldots, X_{t-k}) \end{aligned}$$

Compute the Bollinger bands for your data series and compare the trading strategies: buy when $X_t < BB_t^L$ and sell when $X_t > BB_t^L$, or moment type, that is, buy when $X_t > BB_t^U$ and sell when $X_t < BB_t^L$.

6. Generate data from a random walk with normal increments and from a random walk with Cauchy increments. Graph the resulting time series and comment on its behavior.

7. Determine whether the following processes $\{y_t\}$ are: (a) uncorrelated sequences; (b) martingale difference sequences

(a) $y_t = \varepsilon_t + \theta\varepsilon_{t-1}$, where ε_t is iid with mean zero and finite variance

(b) $y_t = \varepsilon_t\varepsilon_{t-1}$, where ε_t is iid with mean zero and finite variance

(c) $y_t = s_t\varepsilon_t$, where ε_t is iid with mean zero and finite variance and s_t is a deterministic sequence

(d) $y_t = s_t\varepsilon_t$, where ε_t is iid with mean zero and finite variance and s_t is a stochastic sequence with mean one and the process $\{s_t\}$ is independent of the process $\{\varepsilon_t\}$

(e) $y_t = s_t\varepsilon_t$, where ε_t is iid with mean zero and finite variance and s_t is a stochastic sequence that depends only on past values of y, i.e., $y_{t-1}, y_{t-2}, \ldots$

(f) $y_t = 1$ if $\varepsilon_t > 0$ and $y_t = -1$ if $\varepsilon_t \leq 0$, where ε_t is i.i.d. with mean zero.

8. Suppose that we observe $\{Y_1, \ldots, Y_T\}$ from a stationary series. Consider the following alternative estimators of the autocorrelation function:

$$\widehat{\rho}_1(j) = \frac{\sum_{t=j+1}^{T}\left(Y_t - \overline{Y}\right)\left(Y_{t-j} - \overline{Y}\right)}{\sum_{t=j+1}^{T}\left(Y_t - \overline{Y}\right)^2}$$

$$\widehat{\rho}_2(j) = \frac{\sum_{t=j+1}^{T}\left(Y_t - \overline{Y}\right)\left(Y_{t-j} - \overline{Y}\right)}{\sqrt{\sum_{t=j+1}^{T}\left(Y_t - \overline{Y}\right)^2 \sum_{t=j+1}^{T}\left(Y_{t-j} - \overline{Y}\right)^2}}$$

$$\widehat{\rho}_3(j) = \frac{\sum_{t=j+1}^{T}\left(Y_t - \overline{Y}\right)\left(Y_{t-j} - \overline{Y}\right)}{\sum_{t=1}^{T}\left(Y_t - \overline{Y}\right)^2}$$

$$\widehat{\rho}_4(j) = \frac{\frac{1}{T-j}\sum_{t=j+1}^{T}\left(Y_t - \overline{Y}\right)\left(Y_{t-j} - \overline{Y}\right)}{\frac{1}{T}\sum_{t=1}^{T}\left(Y_t - \overline{Y}\right)^2}.$$

Here, $\overline{Y} = \sum_{t=1}^{T} Y_t/T$. Compare the properties of $\widehat{\rho}_h(j)$, $h = 1, 2, 3, 4$.

9. Give an outline argument for the large sample properties of the sample autocovariance

$$\widehat{\gamma}(j) = \frac{1}{T}\sum_{t=j+1}^{T}\left(Y_t - \overline{Y}\right)\left(Y_{t-j} - \overline{Y}\right)$$

under the assumption that Y_t is i.i.d. with finite fourth moment.

10. Consider the $VR(q)$ test with $q = 3$. Find a time series process that is not i.i.d. but for which $VR(3) = 1$. Comment on the properties of the test in this case.

11. The Chinese Zodiac is an annually changing 12 year cycle. The mean and standard deviation of the return (in percentages) on the US market over 1927-2015 divided by Chinese star sign is given below

Statistic	Rabbit	Dragon	Snake	Horse	Goat	Monkey	Rooster	Dog	Pig	Rat	Ox	Tiger
Mean	12.9	8.4	-1.4	5.4	6.0	9.4	9.8	6.2	12.0	5.4	4.6	13.1
Std Deviation	10.5	13.0	13.2	20.8	33.9	5.5	30.8	16.9	19.2	20.6	27.5	19.8

(a) Carry out a test of the null hypothesis that the returns are equal across sign versus the hypothesis that they are different.

(b) How would you carry out a test of the null hypothesis that the Sharpe ratios are the same (what additional information would you need or what additional assumptions).

12. Suppose that you have a panel dataset of stock returns observed at the daily frequency. Let $\widehat{\rho}_{ij}(k)$ denote the sample cross-autocorrelation from i, j at lag $k \neq 0$, that is,

$$\widehat{\rho}_{ij}(k) = \frac{\sum_{t=k+1}^{T}\left(R_{it} - \overline{R}_i\right)\left(R_{j,t-k} - \overline{R}_j\right)}{\sqrt{\sum_{t=k+1}^{T}\left(R_{it} - \overline{R}_i\right)^2 \sum_{t=k+1}^{T}\left(R_{j,t-k} - \overline{R}_j\right)^2}}.$$

Let

$$\tau_{ij} = \widehat{\rho}_{ij}(k) - \widehat{\rho}_{ji}(k).$$

(a) What is the meaning of τ_{ij}?

(b) Suppose that returns are iid. What is the large sample distribution of τ_{ij}?

13. Suppose that you have a cross sections of stock returns observed at the daily frequency. Consider the test statistics

$$\tau(p,k) = \sum_{j=1}^{k}\sum_{i=1}^{p}\widehat{\rho}_i^2(j), \quad VR(p,k) = 1 + \sum_{i=1}^{p}\sum_{j=1}^{k}\left(1 - \frac{j}{k}\right)\widehat{\rho}_i(j)$$

where $\widehat{\rho}_i(k)$ is the sample autocorrelation for the i^{th} stock computed from a sample of size T. Suppose that stock returns R_{it} are i.i.d. with mean zero and variance σ_i^2 across time and are mutually independent.

(a) Show that $\tau(p,k)$ is approximately χ_{pk}^2 distributed for each k and p.

(b) Show that $VR(p,k)$ is approximately normally distributed for each k and p. Calculate the approximate variance.

(c) Suppose now that returns are correlated contemporaneously with

$$\operatorname{cov}(R_{it}, R_{js}) = \begin{cases} \sigma_{ij} \text{ if } t = s \\ 0 \text{ if } t \neq s. \end{cases}$$

What is the large sample distribution of $\tau(p,k)$?

14. For the sample of Dow Jones stocks compute the momentum portfolios weights

$$w_{it}(j) = \frac{1}{n} \left(r_{i,t-j} - \overline{r}_{t-j} \right)$$

for each period t and realize the return at period $t+K$. Do this for each time period t for which $t-j$ and $t+K$ observations are available and report the realized profit over the sample period. What can you say about the profits of the corresponding contrarian portfolio? Now consider the portfolio weights $w_{it} = \sum_{j=1}^{J} w_{it}(j)$ for integer J and calculate its profits.

15. Suppose that

$$p_t = \mu + p_{t-1} + \varepsilon_t,$$

where ε_t is i.i.d. with mean zero and variance σ^2. Suppose that we observe the log prices on non-holidays, e.g., $p_1, \ldots, p_5, p_8, \ldots$. Define the imputed Saturday and Sunday prices as

$$p_6^I = p_5 + \frac{p_8 - p_5}{3} \quad ; \quad p_7^I = p_5 + \frac{2(p_8 - p_5)}{3}.$$

(a) Show that

$$E\left(p_6^I | p_5, \ldots\right) = p_5 + \mu$$

$$E\left(p_7^I | p_5, \ldots\right) = p_5 + 2\mu,$$

which is consistent with the Martingale Difference Sequence (MDS) hypothesis, but that

$$E\left(p_7^I | p_6^I, p_5, \ldots\right) = 2(p_6^I - p_5) + p_5 = 2p_6^I - p_5 \neq p_6^I.$$

16. Suppose that stock returns satisfy

$$E(R_t | \mathcal{F}_{t-1}) = R_{ft} + \pi_t = R_{ft} + \pi(X_{t-1})$$

for some variable X_{t-1} in the market wide information set $\mathcal{F}_{t-1}$, where R_{ft} is the risk free rate. Therefore, write

$$R_t - R_{ft} = \pi(X_{t-1}) + \varepsilon_t,$$

where $E(\varepsilon_t|\mathcal{F}_{t-1}) = 0$. Now suppose that X_t is i.i.d. and the process $\{X_t\}$ is independent of $\{\varepsilon_t\}$.

(a) What does the weak form efficient markets hypothesis say about the autocorrelation of returns?

17. Suppose that

$$R_t = \mu_t(\theta) + \varepsilon_t,$$

where ε_t are i.i.d. with mean zero and finite variance, while $\mu_t(\theta)$ is some nonlinear time varying mean depending smoothly on parameters $\theta \in \mathbb{R}^p$. Suppose that you have an estimator $\widehat{\theta}$ that satisfies

$$\widehat{\theta} - \theta = \frac{1}{T}\sum_{t=1}^{T} \psi_t + \mathfrak{R}_T,$$

where ψ_t are i.i.d. with mean zero and finite variance, and $\sqrt{T}\mathfrak{R}_T \xrightarrow{P} 0$.

(a) What is the limiting distribution of the sample autocorrelation coefficient based on the residuals, i.e.,

$$\widehat{\gamma}(j) = \frac{1}{T}\sum_{t=j+1}^{T} \left(R_t - \mu_t(\widehat{\theta})\right)\left(R_{t-j} - \mu_{t-j}(\widehat{\theta})\right).$$

in this case? State clearly what additional assumptions you need to make.

18. The **semivariogram**, which is widely used in spatial statistics, can also be used to test the EMH. This is defined as follows for each j

$$\widetilde{sv}(j) = \frac{1}{2(T-j)}\sum_{t=j+1}^{T} \left(Y_t - Y_{t-j}\right)^2.$$

(a) Show that this statistic is an unbiased estimator of

$$sv(j) = E\left[\left(Y_t - Y_{t-j}\right)^2\right] = \gamma(0) - \gamma(j)$$

for all j. Under the EMH, $sv(j) = \gamma(0)$ for all j. For example, we may test the hypothesis that $sv(1) - sv(2) = 0$ by looking at $\widetilde{sv}(1) - \widetilde{sv}(2)$.

(b) What is the limiting distribution of this test statistic under **rw1**?

19. Explain what you think of the following statements regarding the Efficient Markets Hypothesis (EMH)

   (a) Although the EMH claims investors cannot outperform the market, analysts such as Warren Buffet have done exactly that. Hence the EMH must be incorrect.

   (b) According to the weak form of the EMH, technical analysis is useless in predicting future stock returns. Yet financial analysts are not driven out of the market, so their services must be useful. Hence, the EMH must be incorrect.

   (c) The EMH must be incorrect because stock prices are constantly fluctuating randomly.

   (d) If the EMH holds, then all investors must be able to collect, analyze and interpret new information to correctly adjust stock prices. However, most investors are not trained financial experts. Therefore, the EMH must be false.

20. Suppose that the Calendar Time hypothesis holds and that we observe closing stock prices P_t at roughly 255 days in the calendar year, which excludes all weekends, and public holidays, which can fall on any day Monday to Friday. Explain how you can carry out variance ratio tests of EMH with this data.

21. Suppose log prices have a permanent/transitory decomposition:

$$p_t = p_t^* + u_t$$

$$p_t^* = \mu + p_{t-1}^* + \varepsilon_t, \varepsilon_t \sim IID(0, \sigma^2)$$

   where u_t is i.i.d. and independent of ε_t, while p_t^* is a random walk plus drift.

   (a) Calculate $VR(q)$ in this case based on the observed returns $r_t = p_t - p_{t-1}$.

22. Consider the following simple Trading Strategy: if $r_t > \overline{r}$ (mean value), then buy one unit, i.e., $Q_t = +1$. If $r_t < \overline{r}$ (mean value), then sell one unit, i.e., $Q_t = -1$. Rebalance every day at the close.

   (a) What is the realized profit per day?

   (b) What is the expected profit per day

i. if returns are iid

ii. if returns are AR(1) process.

(c) Carry out this strategy for the daily returns on the S&P500 index and calculate the profit of this strategy and compare it with the buy and hold strategy.

23. Consider the regression model for daily stock returns

$$R_t = \alpha + \beta^{\intercal} D_t + \varepsilon_t,$$

where $E(\varepsilon_t|D_t) = 0$. Suppose that D_t is a set of daily dummy variables for Monday, Tuesday, Wednesday, Thursday, and Friday, that is, $D_{1t} = 1$ if day t is Monday and zero otherwise, $D_{2t} = 1$ if day t is Tuesday and zero otherwise, etc.

(a) What does the EMH predict about the coefficients β?

(b) Consider now the regressions

$$R_t = \sum_{j=1}^{5} b_j D_{jt} + \varepsilon_t$$

$$R_t = \alpha + \sum_{j=1}^{4} \beta_j D_{jt} + \varepsilon_t.$$

What restrictions does the EMH make on b_j?

24. Consider the following model for daily stock return data

$$R_t = \beta^{\intercal} X_t + \varepsilon_t, \quad t = 1, \ldots, T$$

where X_t contains: daily dummies, monthly dummies and yearly dummies, that is

$$X_t = (D_t^d, D_t^m, D_t^y),\ D_t^d = (D_{1t}^d, \ldots, D_{5t}^d),\ D_t^m = (D_{1t}^m, \ldots, D_{12t}^m),\ D_t^y = (D_{1t}^y, \ldots, D_{nt}^y),$$

where $D_{1t}^d = 1$ if day t is Monday and zero otherwise, $D_{2t}^d = 1$ if day t is Tuesday and zero otherwise etc., $D_{1t}^m = 1$ if day t is in January etc., and $D_{1t}^y = 1$ if day t is in year one of the sample and zero otherwise etc. That is, D_t^d is $T \times 5$, D_t^m is $T \times 12$, and D_t^y is $T \times n$, where n is the number of years in the sample (Peress, J., and D. Schmidt (2020). Glued to the TV: Distracted Noise Traders and Stock Market Liquidity. Journal of Finance forthcoming)

(a) First suppose that $X_t = D_t^d$. What does the matrix $X^\intercal X$ look like? Fit this regression model to daily returns on the S&P500. Likewise with $X_t = D_t^m$ and $X_t = D_t^y$.

(b) Now suppose that $X_t = (D_t^d, D_t^m)$. What does the matrix $X^\intercal X$ look like, i.e., what are the blocks $(D_t^d)^\intercal D_t^d$, $(D_t^d)^\intercal D_t^m$, etc.? What further restrictions are needed to avoid multicollinearity in the regression? Fit the revised regression model to daily returns on the S&P500.

(c) Now suppose that $X_t = (D_t^d, D_t^m, D_t^y)$. What does the matrix $X^\intercal X$ look like? What further restrictions are needed to avoid multicollinearity in the regression? Fit the revised regression model to daily returns on the S&P500.

(d) We may do a theoretical analysis of a regular version of this model with three categories of lengths n_1, n_2, and n_3, where $n_1 < n_2 < n_3$. Theoretically, if $T \to \infty$ then $n/T \to 1/250$. Show that even though the estimates of the β corresponding to D_t^y are inconsistent the estimates corresponding to D_t^d and D_t^m are consistent and that t-tests and Wald tests about these coefficients are consistent.

25. Suppose that you have a sample of monthly stock returns R_t and wish to test whether there is an annual cycle in the series. Consider the regression model

$$R_t = \mu_0 + \mu_1 \sin(2\pi t/12) + \varepsilon_t.$$

(a) What are the meanings of the parameters μ_0, μ_1

(b) Estimate μ_0, μ_1 from monthly data on the Fama French market return and test the hypothesis that $\mu_1 = 0$.

(c) Now consider the model

$$R_t = \mu_0 + \mu_1 \sin(2\pi t/12) + \mu_2 \cos(2\pi t/12) + \varepsilon_t.$$

Compare the two models with regard to their predictions for monthly stock returns. Test the hypothesis that $\mu_1 = \mu_2$.

2 Market Microstructure

1. Suppose that true returns $r_1, \ldots, r_T$ are recorded as

$$0, \ldots, 0, r_1 + \cdots + r_k, 0, \ldots, r_{k+1} + \cdots + r_{2k}, \ldots, 0, \ldots, 0, r_{T+1-k} + \cdots + r_T,$$

where $T = j \times k$. Let $\widetilde{r}_t$ denote the typical member of this sequence. Define:

$$\overline{r} = \frac{1}{T}\sum_{t=1}^{T} r_t, \quad s_r^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \overline{r})^2, \gamma_r(s) = \frac{1}{T-s}\sum_{t=s+1}^{T}(r_t - \overline{r})(r_{t-s} - \overline{r})$$

$$\frac{1}{T}\sum_{t=1}^{T}\widetilde{r}_t, \frac{1}{T-1}\sum_{t=1}^{T}(\widetilde{r}_t - \overline{\widetilde{r}})^2, \frac{1}{T-s}\sum_{t=s+1}^{T}(\widetilde{r}_t - \overline{\widetilde{r}})(\widetilde{r}_{t-s} - \overline{\widetilde{r}}).$$

(a) Compare the two return series. What can you say in general?

(b) What happens when $k = T$ and $j = 1$?

(c) How does this relate to the LM non trading model?

2. Consider the non-trading model of LM where a trade occurs in period t with probability $1 - \pi$. Let d_t^* denote the duration (number of periods) of trading, i.e., if period $t - k$ has no trade but each period $t - k + 1, \ldots, t$ has a trade, then $d_t^* = k$. What are the properties of d_t^*, i.e., what is its mean and variance and evolution process.

3. Consider the following bivariate non-trading model for stocks i and j. Let δ_{it} be the nontrade indicator for stock i, where $\delta_{it} = 1$ means no trade and $\delta_{it} = 0$ means a trade in stock i, where $\Pr(\delta_{it} = 1) = \pi_i$ and suppose further that these random variables are not independent in fact they are perfectly correlated meaning that $\delta_{it} = 1$ if and only if $\delta_{jt} = 1$.

   (a) What does this imply about π_i, π_j?

   (b) Suppose that the true returns have identical unit variances and contemporaneous covariance ρ. What is the contemporaneous covariance of observed returns?

4. Epps (1979) reported results showing that stock return cross correlations decrease as the sampling frequency of the data increases. Calculate the cross autocovariance for daily, weekly, and monthly data from individual stock return data and comment on your findings.

5. Nontrading model with stochastic volatility. Consider the non-trading model of Chapter 2. Suppose that we have a stationary stochastic volatility process σ_t independent of everything and

$$r_t = \sigma_t \varepsilon_t,$$

where ε_t is i.i.d. standard normal.

(a) Show that $E((r_t^O)^2) = \sigma^2$

(b) Derive $\text{cov}((r_t^O)^2, (r_{t+1}^O)^2)$.

6. Suppose that daily log stock prices follow the process

$$p_t = a + p_t^* + \eta_t,$$

where

$$p_t^* = \mu + p_{t-1}^* + \varepsilon_t$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ and $\eta_t = \rho\eta_{t-1} + \zeta_t$ with $\zeta_t \sim N(0, \sigma_\zeta^2)$ and $\rho \in (0, 1)$ (ζ_t and ε_s are independent for all t, s).

(a) What is the interpretation of p_t^* and what are the properties of $r_t^* = p_t^* - p_{t-1}^*$. What is the autocorrelation function of the return series $p_t - p_{t-1}$?

(b) Is the observed stock price process consistent with the empirical evidence on daily returns (as presented in Table 2.4 of Campbell, Lo and Mackinlay?

(c) What are some market microstructure explanations for the finding of negative autocorrelation in daily stock returns data?

7. The Roll model says that fundamental prices satisfy

$$P_t^* = \mu + P_{t-1}^* + \varepsilon_t,$$

where ε_t is an uncorrelated sequence. Observed prices satisfy

$$P_t = P_t^* + \frac{1}{2}Q_t s$$

where Q_t is the ± 1 trade indicator.

(a) Show in the Roll model with $\mu = 0$ and ε_t i.i.d. with mean zero and variance σ_ε^2 that

$$\text{cov}((\Delta P_t)^2, (\Delta P_{t-1})^2) = 0.$$

(b) How would you go about testing this implication of the Roll model?

(c) For the data you obtained in the first exercise, check whether this implication seems reasonable when prices or log prices are used.

8. Extend the Roll model to allow the spread s to vary over time so that

$$P_t = P_t^* + \frac{1}{2} Q_t s_t,$$

where Q_t is as before. Suppose that $s_1, \ldots, s_T$ are i.i.d. independent of $Q_1, \ldots, Q_T$ with mean μ_s and variance σ_s^2.

   (a) Calculate

$$\mathrm{cov}((\Delta P_t, \Delta P_{t-1}) \quad ; \quad \mathrm{cov}((\Delta P_t)^2, (\Delta P_{t-1})^2).$$

   (b) Now suppose that $s_1, \ldots, s_T$ are deterministic. Show that

$$\mathrm{cov}((\Delta P_t, \Delta P_{t-1}) = -\frac{1}{4} s_{t-1}^2$$

   (c) Typically, we expect spreads to widen at the open and the close of a market, what should this say about the predictability of returns during the day?

9. Suppose that fundamental prices satisfy

$$P_t^* = \mu + P_{t-1}^* + \varepsilon_t,$$

where ε_t is i.i.d. with mean zero and variance σ_ε^2. Observed prices satisfy

$$P_t^O = \begin{cases} P_{t-1}^O & \text{if there is no trade at } t \text{ (i.e., } \delta_t = 1) \\ P_t^* + \frac{1}{2} Q_t s & \text{if there is a trade at } t \text{ (i.e., } \delta_t = 0), \end{cases}$$

where Q_t is the ± 1 trade indicator (with $\Pr(Q_t = +1) = 1/2$), and

$$\delta_t = \begin{cases} 1 \text{ (no quote update)} & \text{with probability } \pi \\ 0 \text{ (quote update)} & \text{with probability } 1 - \pi. \end{cases}$$

   (a) Derive the properties of $P_t^O - P_{t-1}^O$.

10. What can explain the finding of negative individual stock autocorrelation and positive portfolio or index autocorrelation?

11. The Roll model assumes that trade directions are uncorrelated with changes in the efficient price, i.e., $\mathrm{cov}(Q_t, \varepsilon_t) = 0$. Suppose that $\mathrm{cov}(Q_t, \varepsilon_t) = \rho$ where $\rho \in (0, 1)$. This reflects the notion that a buy order is associated with an increase in the security value.

    (a) Calculate $\mathrm{cov}(\Delta P_t, \Delta P_{t-1})$ and $\mathrm{var}(\Delta P_t)$.

    (b) Show that the usual Roll model estimate of s is upward biased in this case.

12. Consider the Roll model with $E(Q_t) = \vartheta \neq 0$.

    (a) Calculate $E(\Delta P_t)$, $E\left[(\Delta P_t)^2\right]$, and $E\left[\Delta P_t \Delta P_{t-1}\right]$.

    (b) Therefore determine $\mathrm{corr}(\Delta P_t, \Delta P_{t-1})$ and derive a formula for s^2.

13. Suppose that (logarithmic) returns r_t are observed at the daily frequency. Consider the (forward) aggregated returns, for $K = 1, 2, \ldots$ and for $t = 1, \ldots$

$$r_{t,K} = r_t + r_{t+1} + \ldots + r_{t+K-1}.$$

    Suppose that daily returns are i.i.d. normally distributed with $E(r_t) = \mu$ and $\mathrm{var}(r_t) = \sigma^2$. What are the properties of $r_{t,K}$? In particular, calculate the mean, the variance and the autocovariance function. Suppose I compute the backward aggregation

$$r_{t,-K} = r_t + r_{t-1} + \ldots + r_{t-K+1}.$$

    (a) What is the cross covariation $\mathrm{cov}(r_{t,K}, r_{s,-K})$?

    (b) Suppose that

$$r_{t+1} = \alpha + \beta x_t + \varepsilon_t$$

    where x_t, ε_t are i.i.d. and mutually independent. Then consider the regression

$$r_{t+1,K} = a + b x_{t,K} + e_t$$

    What are the values of a, b and what are the properties of e_t?

14. Consider the model

$$p_t = p_{t-1} - \alpha\left(p_{t-1} - p^*_{t-1}\right) + \varepsilon_t$$

$$p^*_t = p^*_{t-1} + \eta_t.$$

(a) Calculate the autocorrelation function of observed returns.

(b) Relate this to the Roll model.

15. Suppose that the efficient price is affected by the order flow (adverse selection), i.e.,

$$p_t = p_t^* + sQ_t + \varepsilon_t$$

$$p_t^* = p_{t-1}^* + \lambda Q_t + \eta_t,$$

where λ measures the informativeness of the order flow.

(a) What is the bid-ask spread at the time of the time t transaction?

(b) What is $\mathrm{cov}(\Delta p_t, \Delta p_{t-1})$?

(c) What is the Roll estimator of spread estimating in this case?

16. In the hedge fund industry, reported returns are often highly serially correlated. Suppose that true returns r_t are i.i.d. and normally distributed with $E(r_t) = \mu$ and $\mathrm{var}(r_t) = \sigma^2$. Suppose that reported returns r_t^o satisfy

$$r_t^o = \alpha r_t + (1-\alpha) r_{t-1}$$

for some $\alpha \in (1/2, 1)$, that is, firms only report smoothed returns rather than actual returns.

(a) Calculate the properties of r_t^o including the mean, the variance and $\mathrm{cov}(r_t^o, r_{t-s}^o)$.

(b) How could you estimate the true returns from data on observed returns? That is, given a sample $\{r_1^o, \ldots, r_T^o\}$ how would you estimate $\{r_1, \ldots, r_T\}$?

17. Suppose that true returns are i.i.d. normally distributed with $E(r_t) = \mu$ and $\mathrm{var}(r_t) = \sigma^2$. Suppose that reported returns r_t^o satisfy

$$r_t^o = \begin{cases} r_t \text{ if } |r_t| > \alpha \\ 0 \quad \text{ else,} \end{cases}$$

where α is some constant.

(a) What are the properties of r_t^o, i.e., what is $E(r_t^o)$, var(r_t^o) and cov(r_t^o, r_{t-s}^o)? You may use the fact that for a standard normal density ϕ

$$\int_0^a x\phi(x)dx =: \frac{1}{\sqrt{2\pi}}\left(1 - e^{-\frac{1}{2}a^2}\right).$$

If you cant give explicit expressions, say whether the quantity is smaller or larger than the corresponding quantity for the observed series.

18. Glosten Harris model with unbalanced uninformed traders. Suppose that Value V is chosen from the distribution

$$V = \begin{cases} V_H \text{ } with \text{ } prob \text{ } \frac{1}{2} \\ V_L \text{ } with \text{ } prob \text{ } \frac{1}{2} \end{cases}$$

Type of investor is chosen from

$$T = \begin{cases} I \quad withprob \text{ } \mu \\ U \text{ } withprob \text{ } 1-\mu \end{cases}$$

Strategies: If informed (I), buy if value is high V_H and sell if value is low V_L; If uninformed (U), buy with probability q or sell with probability $p = 1 - q$. The dealer observes order flow Let $Q_1, \dots, Q_t$, where $Q_t = +1$ if order is a buy order and $Q_t = -1$ if order is a sell one.

(a) Derive the distribution of value V given $Q_1 = +1, \dots, Q_t = +1$, i.e, what is the probability that $V = V_H$ given that the dealer observes t buy orders in a row?

19. Glosten Harris model with Value according to a Uniform distribution on $[0, T]$, i.e., V has density and cdf: $f_V(V) = 1/T, \quad F_V(V) = V/T$.

(a) Show that

$$A = \begin{cases} T\left(\frac{1+\mu-\sqrt{1-\mu^2}}{2\mu}\right) & \text{if } \mu \neq 0 \\ \frac{T}{2} & \text{if } \mu = 0 \end{cases}$$

$$B = T\frac{\left(\mu - 1 + \sqrt{1-\mu^2}\right)}{2\mu}$$

Then

$$A - B = \frac{\left(1 - \sqrt{1-\mu^2}\right)}{\mu}T$$

(b) What is the probability of nontrading?

(c) How does this compare with the two point version?

20. Glosten Harris model with Value according to a t-distribution with degrees of freedom $v = 2$, i.e., V has density and cdf:

$$f_V(V) = \frac{1}{2\sqrt{2}\left(1+\frac{V^2}{2}\right)^{3/2}}, \quad F_V(V) = \frac{1}{2} + \frac{V}{2\sqrt{2}\left(1+\frac{V^2}{2}\right)^{1/2}}.$$

Note that the variance is infinite in this case.

(a) Show that

$$A = \sqrt{\frac{2\mu^2}{1-\mu^2}}.$$

Likewise $B = -A$ so that

$$A - B = 2\sqrt{\frac{2\mu^2}{1-\mu^2}}$$

(b) What is the probability of nontrading?

(c) How does this compare with the two point version?

21. Suppose that a broker may buy x shares of stock at time 0 at price C per share. Each period $i = 0, 1, 2, \ldots$ there arrives independently n_i customers with Poisson distribution with mean μ who are willing to pay a price P for each unit. Let $\pi_j = \Pr(n = j) = \mu^j \exp(-\mu)/j!$, $j = 0, 1, \ldots$ How many shares should the broker buy?

22. Suppose that you have a time series of daily returns on a stock i that is traded in a different time zone from stock j. Specifically, the trading day for i is the first 1/3 of the day, and the trading day for j is the second third of the day. The final third of the day contains no trading. We observe the closing prices for each asset on their respective "trading days", which we denote by $P_{i1}, P_{i4}, \ldots,$ and $P_{j2}, P_{j5}, \ldots$ We want to calculate the contemporaneous return covariance. We assume that each stock has i.i.d. return and that the contemporaneous covariance between return on stock i and stock j is γ, that is,

$$\mathrm{cov}(P_{it} - P_{i,t-1}, P_{it+s} - P_{i,t+s-1}) = 0$$

$$\text{cov}(P_{it} - P_{i,t-1}, P_{jt} - P_{j,t-1}) = \gamma.$$

(a) Then show that

$$\text{cov}(P_{i4} - P_{i1}, P_{j5} - P_{j2}) = \text{cov}(P_{i4} - P_{i2}, P_{j4} - P_{j2}) = 2\text{cov}(P_{i2} - P_{i1}, P_{j2} - P_{j1})$$

23. Calculate the Amihud illiquidity measure for the S&P500 stock index using daily closing prices and daily volume obtained from Yahoo. Plot the time series and evaluate its time series properties, its mean, variance, autocovariance, trend line etc.

24. Calculate the monthly VWAP for the S&P500. If P_i is the closing price on day i and V_i is the trading volume for day i, then

$$VWAP_t = \frac{\sum_{i \in month_t} P_i V_i}{\sum_{i \in month_t} V_i}.$$

Plot the VWAP and compare it to the price series

3 Event Studies

1. Suppose you want to apply event study methodology to detect insider trading. Explain some of the issues that may be involved. Specifically, what type of data would you need? What event window would you choose? What econometric methods would you use? You may focus on the country that you chose in the first exercise sheet. For comparison, read the article http://economics.stanford.edu/files/Theses/Theses_2002/Wong.pdf

2. Suppose that our model for stock returns is that

$$R_{it} = \mu_i + \varepsilon_{it},$$

where ε_{it} is iid with mean zero and variance σ_i^2. We wish to test the snowflake hypothesis that stocks after year 2000 behaved differently from stocks before 2000. One version is a temporary one year shift, and another version is a permanent shift.

(a) Define the null and alternative hypotheses within the framework of this model.

(b) With a sample of daily stock return estimate the mean returns for the 1990's $\widehat{\mu}_i$ and the sample variance $\widehat{\sigma}_i^2$. Calculate the CAR using the one year window of daily stock returns in year 2000. Test the null hypothesis of no change.

(c) Instead calculate the mean returns from year 2000 for each stock and carry out a test of the hypothesis based on the difference of two means paradigm. Compare the results.

(d) Now consider a longer event window and see how the results change.

3. Suppose that our null model for stock returns is that

$$R_{it} = \mu_t + \varepsilon_{it},$$

where ε_{it} is iid with mean zero and variance σ_t^2. That is, stocks have time varying mean and variance the same for all stocks. We wish to test the hypothesis that large stocks behave differently from small stocks.

(a) Define the alternative hypothesis within this model.

(b) With a sample of daily large and small stock returns (e.g., Fama French portfolios) estimate the mean returns $\widehat{\mu}_t$ and the sample variance $\widehat{\sigma}_t^2$ for each t. Test the null hypothesis of identical means.

(c) Now consider a wider stratification on size and see how the results change.

4. You want to test the hypothesis that the advent of computerized trading badly affected market quality. Suppose that before January 1st 2008 there was no computerized trading and after there was bucketfulls in the FTSE100 market but not in the FTSE250 market. Explain how you might test this hypothesis. Carry this test out using daily data on the two stock indexes.

5. The Dow Jones stock market index is an historically important bellweather of the US stock market. Its components as of January, 2013 are shown below along with the market cap in

billions of dollars and the prices on Jan 2nd 2013 and Jan 2nd 2020.

Name	Cap $b	2013	2020	Name	Cap $b	2013	2020
Alcoa Inc.	9.88	21.60	21.42	JP Morgan	172.43	44.66	141.09
AmEx	66.71	58.75	125.85	Coke	168.91	37.60	54.99
Boeing	58.58	99.07	333.22	McD	90.21	90.12	200.79
Bank of America	130.52	12.03	35.64	MMM	65.99	94.78	180
Caterpillar	62.07	93.50	150.53	Merck	127.59	41.34	41.34
Cisco Systems	108.74	20.34	48.42	MSFT	225.06	27.62	160.62
Chevron	216.27	110.39	121.43	Pfizer	191.03	25.91	39.14
du Pont	42.64	47.04	63.50	Proctor & Gamble	188.91	69.39	123.41
Walt Disney	92.49	51.10	148.20	AT&T	200.11	35.00	38.86
General Electric	222.31	20.52	11.93	Travelers	28.25	72.86	137.51
Home Depot	94.47	63.48	219.66	United Health	53.21	54.54	292.50
HP	29.49	6.82	20.79	United Tech	77.89	84.00	153.14
IBM	219.20	196.35	135.42	Verizon	126.43	44.27	61.05
Intel	105.20	21.38	60.84	Wall Mart	231.02	69.24	119.94
Johnson[2]	198.28	70.84	145.97	Exxon Mobil	405.60	88.71	70.90

We also show the components as of January 2020 along with their market cap. For comparison, US GDP in 2012 was $tr16.197 and in 2019 $tr21.44[1]

Name	Cap $b	2013	2020	Name	Cap $b	2013	2020
Apple	1300	78.43	300.35	JP Morgan	425		
AmEx	102			Coke	234.3		
Boeing	187.3			McD	157		
Goldman Sachs	87	131.66	234.32	MMM	101.8		
Caterpillar	82.1			Merck	221		
Cisco Systems	200			MSFT	1000		
Chevron	217			Pfizer	219		
Dow chemical	39			Proctor & Gamble	309		
Walt Disney	261			Nike	158	25.92	120.20
Walgreen	45			Travelers	35		
Home Depot	245			United Health	284		
Visa	438	38.85	191.12	United Tech	132		
IBM	122			Verizon	242		
Intel	255			Wall Mart	325		
Johnson[2]	385			Exxon Mobil	287		

Suppose you want to test whether being admitted to the index has a positive effect on your performance and being delisted has a negative effect. We can divide the stocks into three categories: Remainers (in Dow in 2013 and 2020), Leavers (in Dow in 2013 but not in 2020), and Joiners (in Dow in 2020 but not in 2013). 2013).

(a) Given the stock prices of the three groups of stocks reported above calculate their buy and hold returns over the period 2013-2020 and carry out a test of the hypothesis that

[1]On September 20, 2013, Goldman Sachs, Nike, Inc., and Visa Inc. replaced Alcoa, Bank of America, and Hewlett-Packard. Visa replaced Hewlett-Packard because of the split into HP Inc. and Hewlett Packard Enterprise. On March 19, 2015, Apple Inc. replaced AT&T, which had been a component of the DJIA since November 1916. Apple became the fourth company traded on the NASDAQ to be part of the Dow. On September 1, 2017, DowDuPont replaced DuPont. DowDuPont was formed by the merger of Dow Chemical Company with DuPont. On June 26, 2018, Walgreens Boots Alliance replaced General Electric, which had been a component of the DJIA since November 1907, after being part of the inaugural index in May 1896 and much of the 1896 to 1907 period. On April 2, 2019, Dow Inc. replaced DowDuPont. Dow, Inc. is a spin-off of DowDuPont, itself a merger of Dow Chemical Company and DuPont.

$\mu_{joiners} = \mu_{leavers} = \mu_{remainers}$. You may assume that stock returns are exactly normally distributed.

(b) Then consider the market model

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it},$$

where R_{mt} is the return on the market portfolio. Estimate this model for the Dow Jones daily stock returns and compare the values of $\widehat{\alpha}_i$ estimated for the three groups of stocks.

6. Suppose that stock returns R_{it} obey the following market model

$$R_{it} = \alpha_i + \beta_i R_{mt} + \delta_i 1\left(t = t_0\right) + \varepsilon_{it},$$

where $t = 1, \ldots, t_0 - 1, t_0, t_0 + 1, \ldots T$, $i = 1, \ldots, N$, and ε_{it} obey the usual regression assumptions.

(a) What do the parameters $\{\delta_i,\ i = 1, \ldots, N\}$ measure?

(b) How could you test the null hypothesis that $\delta_1 = \delta_2 = \cdots = \delta_N = 0$?

(c) How could you test the null hypothesis that $\delta_1 = \delta_2 = \cdots = \delta_N$?

7. Suppose that for $R_t = (R_{1t}, \ldots, R_{nt})^\intercal$

$$R_t = \Theta x_t + \varepsilon_t,$$

where $x_t = (1, R_{mt})^\intercal$, and $t = 1, \ldots, T$. Suppose that $\varepsilon_t \sim N(0, \Omega_\varepsilon)$ independent of $X = (x_1^\intercal, \ldots, x_T^\intercal)^\intercal$. The OLS estimator of Θ, where $\Theta^\intercal = (\theta_1, \ldots, \theta_n)$ with $\theta_i = (\alpha_i, \beta_i)^\intercal$, is

$$\widehat{\Theta} = \sum_{t=1}^{T} R_t x_t^\intercal \left(\sum_{t=1}^{T} x_t x_t^\intercal \right)^{-1}.$$

Suppose that we observe the returns over the event window $T + 1, \ldots, T + \tau$, and let $\widehat{\varepsilon}_{T+j} = R_{T+j} - \widehat{\Theta} x_{T+j}$, $j = 1, \ldots, \tau$, and let $\widehat{\varepsilon}^* = (\widehat{\varepsilon}_{T+1}, \ldots, \widehat{\varepsilon}_{T+\tau})^\intercal \in \mathbb{R}^{n\tau}$.

(a) Show that under the null hypothesis of no effect (i.e., no change between $\{1, \ldots, T\}$ and $\{T + 1, \ldots, T + \tau\}$

$$\widehat{\varepsilon}^* \sim N(0, V)$$

$$V = \left(I_\tau + \frac{1}{T} M_{T,\tau}\right) \otimes \Omega_\varepsilon$$

$$M_{T,\tau} = \left(\frac{1}{T} x_{T+j}^\intercal \left(\frac{1}{T} \sum_{t=1}^{T} x_t x_t^\intercal \right)^{-1} x_{T+k} \right)_{j,k} .$$

Here, $\otimes$ denotes the matrix Kronecker product.

(b) Define the average CAR

$$\widehat{CAR}(\tau) = \frac{1}{n} \sum_{i=1}^{n} \widehat{CAR}_i(\tau) = \frac{1}{n\tau} \sum_{i=1}^{n} \sum_{s=1}^{\tau} \widehat{\varepsilon}_{i,T+s}.$$

Show how to carry out the test of no effect

i. When T is large

ii. When T is not so large

(c) Define the Wald statistic

$$W = \widehat{\varepsilon}^{*\intercal} \widehat{V}^{-1} \widehat{\varepsilon}^*,$$

where $\widehat{V}$ is a consistent estimator of V. Compare this test with the CAR based test.

4 Market Model, CAPM, APT

1. For the data you selected in problem 1,

(a) Using the full sample regress the excess returns of the individual stocks on the index return and perform tests that the intercept is zero. Report the point estimates, t-statistics, and whether or not you reject the CAPM. Compare the results according to whether you use the iid standard errors or the heteroskedasticity consistent ones.

(b) For each stock perform the same test over each of two subsamples of equal size, and report the point estimates, t-statistics, and whether or not you reject the CAPM in each subperiod

(c) Perform joint tests of the CAPM using both stocks using the F-test statistic for the whole period and each subperiod.

2. Suppose that R is a random variable with mean μ and variance σ^2, while R_f is a non random risk free rate. Consider the portfolio

$$R_p(w) = wR + (1-w)R_f$$

for $w \in \mathbb{R}$.

   (a) Calculate the mean and variance of $R_p(w)$ as a function of μ, σ^2, and R_f

   (b) Determine the optimal choice of w that maximizes the mean variance utility function

   $$U(w) = m(w) - \gamma s^2(w),$$

   where $m(w) = E(R_p(w))$ and $s^2(w) = \mathrm{var}(R_p(w))$ and γ is a risk aversion parameter.

3. Suppose that X and Y are mean μ random variables with $\mathrm{var}X = \sigma_X^2$ and $\mathrm{var}Y = \sigma_Y^2$ and suppose that $\mathrm{cov}(X,Y) = \sigma_{XY} = \sigma_X\sigma_Y\rho_{XY}$ with $|\rho_{XY}| \leq 1$. You invest a fraction ω of your wealth in X and $1-\omega$ in Y, called portfolio $P(\omega)$. Show that:

   (a) (no short sales case) For all $\omega \in [0,1]$

   $$\mathrm{var}(P(\omega)) \leq \max\{\sigma_X^2, \sigma_Y^2\}$$

   (b) Now allow $\omega \in \mathbb{R}$. Show that the optimal ω satisfies

   $$\omega_{opt} = \frac{\sigma_Y^2 - \sigma_X\sigma_Y\rho_{XY}}{\mathrm{var}(X-Y)}$$

   (provided $\mathrm{var}(X-Y) > 0$), and for this value

   $$\mathrm{var}(P(\omega_{opt})) \leq \min\{\sigma_X^2, \sigma_Y^2\}$$

   Under what conditions would ω be negative?

4. A set $S \in \mathbb{R}^n$ is convex if

$$\lambda x + (1-\lambda)y \in S, \quad x, y \in S, \lambda \in [0,1].$$

   (a) Show that the set of mean variance efficient portfolios is convex.

5. Suppose that the $n \times n$ covariance matrix of stock returns satisfies

$$\Sigma = bb^{\intercal} + \sigma^2 I_n,$$

where $b = (b_1, \ldots, b_n)^{\intercal}$.

(a) What is the variance of the equally weighted portfolio?

(b) What is the variance of the global minimum variance portfolio?

i. In the case that b is a vector of random variables from the uniform on $[0, 1]$ distribution

ii. In the case that $b = \beta i_n$ for some positive scalar β.

6. Suppose that you have initial wealth W_0 that you can divide into a risky asset with payoff $\mathcal{R} \geq 0$ per dollar invested and a riskless asset that pays off $R_f \geq 1$ per dollar invested. You choose a fraction $\omega \in [0, 1]$ to invest in the risky asset. The wealth next period is

$$W_1 = W_0 \left(\omega \mathcal{R} + (1 - \omega) R_f \right).$$

The payoff at period n if the same proportion is invested every period and if the risky asset payoffs are iid is

$$W_n = W_0 \prod_{i=1}^{n} \left(\omega \mathcal{R}_i + (1 - \omega) R_f \right).$$

Suppose that $W_0 = 1$ and that utility is logarithmic, that is, you see to choose ω to maximize $E\left(\log W_n\right)$. Discuss the optimal choice of ω in the case where n is large.

7. The information ratio of a security with payoff R relative to a benchmark with random payoff R_b is

$$IR = \frac{E\left(R - R_b\right)}{\sqrt{\mathrm{var}(R - R_b)}}.$$

(a) Given data $\{R_t, R_{bt}, t = 1, \ldots, T\}$ how would you estimate the information ratio.

(b) How would you test the hypothesis that $IR = 0$ versus the hypothesis that $IR > 0$. You may assume that the data are i.i.d.

8. Find enclosed daily data (from 19630701 to 20181231) on the Fama French five factors, MARKET, HML, SMB, RMW, and CMA, and the risk free rate Rf, where MARKET is the excess

return (relative to the risk free rate) on the CRSP value weighted market portfolio, HML, SMB, RMW, and CMA are the returns on zero net investment portfolios as described on Ken French's website. The calculations can be performed in Excel and/or Eviews, but also in other software packages, as you prefer.

(a) Compute the sample statistics of all the stock return series and the risk free rate series, i.e., the mean, standard deviation, skewness and kurtosis. Are these return series approximately normal?

(b) Compute the first 20 autocorrelation coefficients for all the series and test whether the series is linearly predictable or not using the acf test, the Box-Pierce, and the variance ratio. Comment on your results.

(c) Now divide the time period into decades and repeat the analysis from part b. Comment on your results.

9. The Fama French portfolios (HML, SMB, RMW, and CMA) are zero net investment portfolios. What should be their expected returns? Define a test of this hypothesis and carry it out on the four FF portfolios. Comment on your results.

10. Suppose that

$$E_i = \gamma_0 + \beta_i \gamma_1 + \beta_i^3 \gamma_2,$$

where $E_i = E(R_{it} - R_{ft})$ and

$$\beta_i = \frac{\mathrm{cov}(R_{it} - R_{ft}, R_{mt} - R_{ft})}{\mathrm{var}(R_{mt} - R_{ft})}.$$

Here, R_{it} are the returns on asset i at time t, R_{ft} is the risk free rate known at time $t-1$, and R_{mt} is the return on the market portfolio at time t.

(a) What implications does the CAPM make about $\gamma_0, \gamma_1, \gamma_2$?

(b) Explain why

$$\beta_i = \frac{\mathrm{cov}(R_{it}, R_{mt})}{\mathrm{var}(R_{mt})}.$$

(c) You want to test the CAPM with the data $\{R_{it}, R_{mt}, R_{ft},\ t = 1, \ldots, T,\ i = 1, \ldots, n\}$. Here, n is large and T is large. Describe the Fama-Macbeth methodology for carrying out this test.

(d) Will this test work in this case when the risk free rate is not observed but an unbiased proxy R^*_{ft} is available that satisfies $E(R^*_{ft}) = R_{ft}$? Or rather how should you modify the test to make it work? (Hint, you can define

$$\widehat{E}^*_i = \frac{1}{T}\sum_{t=1}^{T}\left(R_{it} - R^*_{ft}\right),$$

which is a consistent estimator of E_i.)

(e) What is an alternative way of testing the CAPM when there is no risk free rate?

11. Suppose that the $n \times n$ covariance matrix satisfies

$$\Omega = B\Sigma B^{\intercal}$$

where the $n \times K$ matrix B is of rank $K \leq n$ and $\Sigma = I_K$.

(a) Show that there exists an $n \times K$ orthonormal matrix B^* with $B^* B^{*\intercal} = I_n$ and a diagonal matrix Σ^* with

$$\Omega = B^*\Sigma^* B^{*\intercal}.$$

(b) Show the converse.

12. Consider the market model

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it},$$

where ε_{it} are iid with mean zero and variance σ_i^2, while R_{mt} is iid with mean μ_m and variance σ_m^2.

(a) Show that the $n \times n$ covariance matrix of returns $R_t = (R_{1t}, \ldots, R_{nt})^{\intercal}$ satisfies

$$\Omega = \sigma_m^2 \beta\beta^{\intercal} + D$$

for some diagonal matrix D.

(b) What is the conditional variance matrix of R_t given R_{mt}?

13. Suppose that the fundamental price P^* satisfies

$$P^*_{it} = P^*_{it-1} + \varepsilon_{it},$$

where ε_{it} are i.i.d. with mean zero across both i and t. Buy and sell orders arrive randomly. The full spread is s_i and the half-spread is $s_i/2$. We have for each firm

$$P_{it} = P_{it}^* + Q_{it}\frac{s_i}{2},$$

where Q_{it} is a trade direction indicator, +1 for buy and -1 if customer is selling. Assume that Q_{it} is i.i.d. across i and t with equal probability of +1 and -1 and unrelated to P_{it}^*. Suppose that one considers a portfolio with weights $\{w_i\}_{i=1}^n$ and let $P_t^w = \sum_{i=1}^n w_i P_{it}$ denote the value of the portfolio at time t. Obtain an expression for the autocorrelation of P_t^w in two cases

(a) The portfolio is well diversified, i.e., $w_i = 1/n$ with n very large

(b) The portfolio consists of a large position in the first asset and a diversified position across the remaining large set of assets, i.e., $w_1 = 0.5$ and $w_j = 1/2n$ for $j = 2, 3, \ldots$

14. In the country of Dupostan there are only 2 securities whose returns R_{1t} and R_{2t} are observed over time periods $t = 1, \ldots, T$. The market portfolio is formed from the equal mixture of 1 and 2, i.e., $R_{mt} = 0.5R_{1t} + 0.5R_{2t}$. The risk free rate is exactly zero. Describe how you would test the CAPM in this world.

15. What role does the assumption of normality play in testing the Capital Asset Pricing Model? What is the evidence regarding normality in stock returns? If stock returns are not normal and indeed have heavy tailed distributions with some extreme outliers, what are the properties of the standard normal-based tests of this hypothesis?

16. Suppose that stock returns satisfy the market model

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it},$$

and no risk free rate is observed.

(a) Suppose that $n = 2$. Find unit cost portfolio weights w_1, w_2 that define a market neutral portfolio in terms of β_1, β_2.

(b) Suppose that $n = 3$. Find unit cost portfolio weights w_1, w_2, w_3 that define a market neutral portfolio in terms of $\beta_1, \beta_2, \beta_3$.

(c) The zero beta portfolio is defined as the market neutral portfolio of assets $i = 1, \ldots, n$ with minimum variance. Write down the Lagrangean for this problem and solve for the weights of the zero beta portfolio.

(d) Explain why this is the same as the frontier portfolio (mean variance efficient) that is uncorrelated with the market portfolio.

(e) Given a sample of suitable data how would you estimate the weights of the zero beta portfolio?

17. Suppose that

$$Z_{it} = \beta_i Z_{mt} + \varepsilon_{it},$$

where the usual assumptions apply. Now suppose that we estimate

$$\widehat{\beta}_i = \frac{\sum_{t=1}^T Z_{mt} Z_{it}}{\sum_{t=1}^T Z_{mt}^2}$$

for $i = 1, \ldots, N$. Suppose we just work with the two extreme assets according to their estimated betas. Let $\widehat{\beta}_{\max} = \widehat{\beta}_{i_{\max}}$ and $\widehat{\beta}_{\min} = \widehat{\beta}_{i_{\min}}$, where

$$i_{\max} = \arg \max_{1 \le i \le N} \widehat{\beta}_i \quad ; \quad i_{\min} = \arg \min_{1 \le i \le N} \widehat{\beta}_i.$$

Define the estimated risk premium from the cross-sectional regression of excess returns

$$\widehat{\gamma} = \frac{\overline{Z}_{i_{\max}} \widehat{\beta}_{\max} + \overline{Z}_{i_{\min}} \widehat{\beta}_{\min}}{\widehat{\beta}_{\max}^2 + \widehat{\beta}_{\min}^2}.$$

Investigate the performance of $\widehat{\gamma}$ for simulated data. Specifically, suppose that $\beta_i \sim U[0, 2]$, while ε_{it}, Z_{mt} are all standard normal and independent of each other. Consider different size T, N.

18. The **Durbin-Wu-Hausman** test is based on comparing two estimators. Consider the market model and define the vector of unrestricted and restricted estimators of β, $\widehat{\beta}$ and $\widetilde{\beta}$ respectively.

(a) Construct a test of the CAPM, that $\alpha = 0$ using this approach.

(b) Does this test have power against all alternatives?

19. Suppose that asset returns satisfy

$$R_{it} - E(R_{it}) = f_t b_i + \varepsilon_{it}$$

where ε_{it} are i.i.d. with mean zero and variance σ_ε^2. The time series of scalar factors $\{f_t\}$ and the cross section of scalar loadings $\{b_i\}$ are unobserved. Calculate the two matrices Ω, Σ with typical elements

$$\Omega_{ij} = \frac{1}{T}\sum_{t=1}^{T}\left(R_{it} - E(R_{it})\right)\left(R_{jt} - E(R_{jt})\right)$$

$$\Sigma_{ts} = \frac{1}{n}\sum_{i=1}^{n}\left(R_{it} - E(R_{it})\right)\left(R_{is} - E(R_{is})\right)$$

Suppose that f_t are i.i.d. $N(0,1)$ and b_i are i.i.d. $N(0,1)$ and both processes are independent of all of ε.

(a) Obtain the probability limit of Ω as T gets big

(b) Obtain the probability limit of Σ as n gets big.

20. Suppose that

$$R_{it} - R_{ft} = \alpha_i + \beta_i\left(R_{mt} - R_{ft}\right) + \varepsilon_{it},$$

where we don't observe R_{mt} but we observe a proxy f_t that obeys

$$f_t = \pi_0 + \pi_1(R_{mt} - R_{ft}) + \eta_t,$$

where η_t is mean zero given all the right hand side variables.

(a) How can one test the CAPM ($\alpha_i = 0$) in this case?

(b) Suppose that the risk free rate is not observed and instead

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}.$$

How can one test the Black version of the CAPM (that $\alpha_i = (1 - \beta_i)\gamma$ for some γ) in this case?

21. The Sharpe-Lintner CAPM predicts that

$$\mathrm{var}(R_{mt})E(R_{it} - R_{ft}) - \mathrm{cov}(R_{it} - R_{ft}, R_{mt} - R_{ft})E(R_{mt} - R_{ft}) = 0$$

for each asset i. Provide a test of this restriction using the estimated quantities:

$$\widehat{\mathrm{var}}(R_{mt}) = \frac{1}{T-1}\sum_{t=1}^{T}\left(R_{mt} - \overline{R}_m\right)^2$$

$$\widehat{E}(R_{it} - R_{ft}) = \frac{1}{T}\sum_{t=1}^{T}(R_{it} - R_{ft})$$

$$\widehat{\mathrm{cov}}(R_{it} - R_{ft}, R_{mt} - R_{ft}) = \frac{1}{T-1}\sum_{t=1}^{T}\left(R_{it} - \overline{R}_i\right)\left(R_{mt} - \overline{R}_m\right)$$

$$\widehat{E}(R_{mt} - R_{ft}) = \frac{1}{T}\sum_{t=1}^{T}(R_{mt} - R_{ft}),$$

where $\overline{R}_m = \sum_{t=1}^{T} R_{mt}/T$ and $\overline{R}_i = \sum_{t=1}^{T} R_{it}/T$.

22. Suppose that

$$y_{it} = \mu + u_{it},\ i = 1,\ldots,N,\ t = 1,\ldots,T$$

where $E(u_{it}) = 0$. Suppose that

$$u_{it} = \theta_i^{\intercal} f_t + \varepsilon_{it},$$

where ε_{it} is i.i.d. and θ_i and f_t are also i.i.d. random variables with mean zero. Calculate the covariance matrix Σ of the $NT \times 1$ vector $u = (u_{11},\ldots,u_{NT})^{\intercal}$. What is the covariance matrix of the $N \times 1$ and $T \times 1$ vectors

$$\overline{u}_{time} = \left(\sum_{t=1}^{T} u_{1t},\ldots,\sum_{t=1}^{T} u_{Nt}\right)^{\intercal} \quad ; \quad \overline{u}_{cross} = \left(\sum_{i=1}^{N} u_{i1},\ldots,\sum_{i=1}^{N} u_{iT}\right)^{\intercal}.$$

23. Suppose that for $t = 1,\ldots,T$ and $i = 1,\ldots,N$

$$R_{it} = f_t + \varepsilon_{it},$$

where f_t are iid with mean zero and variance σ_f^2, while ε_{it} are iid with mean zero and variance σ_ε^2, and f, ε are mutually independent.

(a) Write this model in vector form for the $N \times 1$ vector $R_t = (R_{1t},\ldots,R_{Nt})^{\intercal}$. Thereby calculate the $N \times N$ matrix

$$\Sigma = E(R_t R_t^{\intercal})$$

in terms of σ_f^2 and σ_ε^2. Why is this model called the equicorrelated case?

(b) What are the eigenvalues and eigenvectors of this matrix and what is Σ^{-1}

(c) Discuss estimation of σ_f^2 and σ_ε^2 when N is fixed and T is large

(d) Write this model in vector form for the $T\times 1$ vector $R_i = (R_{i1}, \ldots, R_{iT})^\intercal$. Thereby calculate the $T \times T$ matrix

$$\Psi = E(R_i R_i^\intercal)$$

in terms of σ_f^2 and σ_ε^2.

(e) Discuss estimation of σ_f^2 and σ_ε^2 when T is fixed and N is large.

5 Time varying Expected returns and Fundamentals v Bubbles

1. Consider the following rational expectations model for stock returns

$$r_t = \alpha + \beta E\left(r_{t+k}|\mathcal{F}_{t-1}\right) + \gamma x_t + \varepsilon_t,$$

where $E\left(\varepsilon_t|\mathcal{F}_{t-1}\right) = 0$ and $x_t = \rho x_{t-1} + \eta_t$, where $E\left(\eta_t|\mathcal{F}_{t-1}\right) = 0$. Suppose that $|\beta|, |\rho| < 1$.

(a) In the case where $k = 0$, show that

$$r_t = \frac{\alpha}{1-\beta} + \frac{\beta\gamma\rho}{1-\beta} x_{t-1} + \gamma x_t + \varepsilon_t$$

(b) In the case where $k = 1$, show that

$$r_t = \frac{\alpha}{1-\beta} + \frac{\gamma}{1-\beta\rho} x_t + \varepsilon_t$$

2. Work with the monthly Shiller data ie_data.xls. from 1878-2019.12. Calculate the monthly series of K period returns by

$$R_i(K) = \frac{P_{i+K} - P_i}{P_i}, \; i = 1, \ldots, n-K,$$

where n is the sample size 1788 monthly observations. Then compute the dividend price ratio D_i/P_i and the earnings price ratio E_i/P_i.

(a) For $K = 1, 12, 24, 36, 48, 60$ plot $R_i(K)$ against D_i/P_i and against E_i/P_i. Comment on the relationship

(b) Calculate the sample acf for D_i/P_i and E_i/P_i and for $R_i(K)$ for $K = 1, 12, 24, 36, 48, 60$ and comment on the results

(c) Now run the regression

$$R_i(K) = \alpha + \beta \frac{D_i}{P_i} + \gamma \frac{E_i}{P_i} + \delta \frac{1}{P_i} + \varepsilon_i$$

using the full sample and comment on the magnitude of the coefficients and their statistical significance. Calculate the acf of the residuals from this regression.

(d) Make a forecast for $R_n(12)$ based on D_n/P_n, E_n/P_n, and $1/P_n$. Bet the house on it.

(e) Now consider the regression

$$R_i(K) = \alpha + \beta \frac{D_i}{P_i} + \gamma \frac{D_{i-1}}{P_{i-1}} + \varepsilon_i$$

3. Slowly varying expected returns. Suppose that

$$r_{t+1} = \mu + x_t + \varepsilon_{t+1},$$

where

$$x_{t+1} = \phi x_t + \xi_{t+1}, \quad -1 < \phi < 1$$

and ε_t, ξ_s are mutually independent for all t, s and are individually i.i.d. with mean zero and variances σ_ε^2 and σ_ξ^2.

(a) Calculate $E(r_t), \mathrm{var}(r_t)$, $E_t(r_{t+1})$, and $\mathrm{var}_t(r_{t+1})$.

(b) Compute the unconditional autocorrelation function

$$\rho(k) = \frac{\mathrm{cov}(r_t, r_{t-k})}{\mathrm{var}(r_t)}$$

(c) Is this consistent with the empirical evidence regarding autocorrelation of return series? What about the evidence for the autocorrelation of squared returns?

4. Suppose that expected returns are a constant plus AR(1) process

$$\mu_t = E_t\left[r_{t+1}\right] = r + x_t$$

$$x_{t+1} = \phi x_t + \xi_{t+1}, \quad -1 < \phi < 1.$$

Suppose also that log dividends follow a random walk, i.e.,

$$d_{t+1} = m + d_t + v_t,$$

where v_t are i.i.d. with mean zero and variance σ_v^2.

(a) Show that

$$d_t - p_t = c + \frac{x_t}{1 - \rho\phi}.$$

(b) How could you test this hypothesis?

5. It is often claimed that using overlapping data like in predictive regression does not add bias, for example, Cochrane (See Problem set 1, Q2, Business 35905 on website): "No, neither coefficients nor R^2 are affected by overlapping data". It is the purpose of this question to explore these claims. Suppose that

$$r_{t+1} = \beta x_t + \varepsilon_{t+1}$$

where $E(\varepsilon_{t+1}|\mathcal{F}_t) = 0$ and x_t is a dynamic process

$$x_{t+1} = \phi x_t + \eta_{t+1},$$

where $E(\eta_{t+1}|\mathcal{F}_t) = 0$. Suppose that ε_t and η_s are mutually independent for all t, s. Define the OLS estimator

$$\widehat{\beta} = \left(\sum_{t=1}^{T-1} x_t^2\right)^{-1} \sum_{t=1}^{T-1} x_t r_{t+1}$$

Define the aggregated returns $r_{t+1:t+K} = r_{t+1} + \ldots + r_{t+K}$ for each L. Define the overlapping OLS estimator

$$\widehat{\beta}(K) = \left(\sum_{t=1}^{T-K} x_t^2\right)^{-1} \sum_{t=1}^{T-K} x_t r_{t+1:t+K} \tag{1}$$

for each $K = 1, 2, \ldots$

(a) Write this model as

$$r_{t+1:t+K} = \beta(K) x_t + u_{t+1:t+K},$$

and give expressions for $\beta(K)$ and $u_{t:t+K} = \varepsilon_{t+1} + \ldots + \varepsilon_{t+K} + \beta\eta_{t+1} + \cdots + \beta\phi^{K-1}\eta_{t+K-1}$.

(b) Argue that $E(u_{t+1:t+K}|\mathcal{F}_t) = 0$.

(c) Suppose that (x_t, ε_t) is a stationary weak dependent process and let $M = E(x_t^2)$.

i. Show that $E(\widehat{\beta}) = \beta$

ii. Show that

$$E(\widehat{\beta}(K)) = \beta(K) - \frac{1}{T} \frac{\sum_{j=1}^{\infty} E\left(x_{t+j}^2 x_t u_{t+1:t+K}\right)}{E^2(x_t^2)} + O(T^{-2}).$$

iii. Show that typically $E\left(x_{t+j}^2 x_t u_{t+1:t+K}\right) > 0$.

6. Blanchard and Watson (1982) model. Suppose that

$$B_{t+1} = \begin{cases} \frac{1+R}{\pi} B_t + \eta_{t+1} & \text{with probability } \pi \\ \eta_{t+1} & \text{with probability } 1-\pi \end{cases}$$

where η_t is i.i.d. with mean zero and variance one.

   (a) What are the properties of the bubble process? What is $E_t(B_{t+1})$? What is $\mathrm{var}_t(B_{t+1})$?

   (b) What is the chance that the bubble lasts for more than 5 periods?

   (c) Suppose that we observed prices that satisfy

$$P_t = P_t^* + B_t$$

$$P_t^* = P_{t-1}^* + u_t$$

   where u_t is normally distributed with mean zero and variance one and u_t and η_t are mutually independent. Show that P_t is a martingale process when $R = 0$.

7. Suppose that a firm pays out dividends D_t every year and its stock price at time t satisfies

$$P_t = E_t\left[\frac{P_{t+1} + D_{t+1}}{1+R}\right] = \frac{E_t P_{t+1}}{1+R} + \frac{E_t D_{t+1}}{1+R},$$

where R is the positive discount rate. Show that the price satisfies

$$P_t = P_t^* + B_t$$

where P_t^* is the "fundamental price" $P_t^* = \sum_{i=1}^{\infty} \left(\frac{1}{1+R}\right)^i E_t D_{t+i}$ and B_t is any stochastic process that satisfies

$$B_t = \frac{E_t B_{t+1}}{1+R}.$$

Consider the following bubble process

$$B_{t+1} = (1+R) B_t + \eta_{t+1},$$

where η_t is independent and identically distributed (iid) with distribution $N(0, \sigma_\eta^2)$.

(a) What are the time series properties of the bubble process?

(b) Suppose that the dividend process D_t is also i.i.d. with positive mean μ, what are the properties of the fundamental price P_t^* and the actual price $P_t = P_t^* + B_t$?

(c) Discuss the plausibility of this model.

(d) Given data on P_t, D_t do you think it would be possible to distinguish this case from the case without the rational bubble, i.e., the case where $B_t = 0$ for all time?

8. Suppose that a firm pays out dividends D_t every year and define the perfect foresight price at time t satisfies
$$P_t^* = \sum_{i=0}^{\infty} \left(\frac{1}{1+R} \right)^i D_{t+i}.$$
Let J_t be some information set that may be smaller than the agent's information set I_t, and let
$$\overline{P}_t = E\left(P_t^* | J_t\right).$$
Recall that $P_t = E\left(P_t^* | I_t\right)$.

(a) Show that
$$\mathrm{var}(P_t) \geq \mathrm{var}(\overline{P}_t)$$

(b) Show that
$$\mathrm{var}(P_t^* - \overline{P}_t) \geq \mathrm{var}(P_t^* - P_t)$$

9. Compare the following two models for stock prices using the S&P500 daily data
$$\log(P_t) = \alpha + \beta t + \varepsilon_t$$
$$\log(P_t) = \alpha + \beta \log(P_{t-1}) + \varepsilon_t.$$
That is, estimate the parameters α, β in each model and evaluate graphically the residuals from each regression.

10. Suppose that the return to holding painting i in period t is
$$r_{it}^o = \mu_t + \varepsilon_{it},$$
where μ_t is the common component and ε_{it} is an error term that is i.i.d. across time with mean zero and variance σ_i^2. Suppose that painting i is bought at time t_{bi} and sold at time t_{si} with

$t_{bi} < t_{si}$ and the prices are only observed at these times. The holding return on painting i (assuming it was bought and sold exactly once) is

$$r_i = \sum_{t=t_{bi}}^{t=t_{si}} \mu_t + \sum_{t=t_{bi}}^{t=t_{si}} \varepsilon_{it}, \; i = 1, \ldots, n.$$

(a) Therefore write

$$r = A\mu + u,$$

where r is the $n \times 1$ vector containing $r_1, \ldots, r_n$ and A is a known $n \times T$ matrix of zeros and one, while u is an $n \times 1$ vector of error terms.

(b) Thereby, show how to estimate μ_t, $t = 1, \ldots, T$ when $n > T$.

11. The dividend yield for the Dow Jones stocks in 2013 and in 2020 are shown in the tables below. Calculate the correlation and rank correlation between the 2013 values and the 2020 values (where both are available) and comment on your results.

	$\%D/P$		$\%D/P$
Alcoa Inc.	1.39	JP Morgan	2.86
AmEx	1.30	Coke	2.94
Boeing	2.65	McD	3.35
Bank of America	0.67	MMM	2.50
Caterpillar	2.36	Merck	4.18
Cisco Systems	2.86	MSFT	3.57
Chevron	3.38	Pfizer	3.59
du Pont	3.66	P&Gamble	3.08
Walt Disney	1.71	AT&T	5.12
General Electric	3.39	Travelers	2.44
Home Depot	1.98	United Health	1.73
HP	3.14	United Tech	2.46
IBM	1.89	Verizon	4.71
Intel	4.47	Wall Mart	2.51
Johnson[2]	3.36	Exxon Mobil	2.76

Name	D/P%	Name	D/P%
Apple	0.98	JP Morgan	2.65
AmEx	1.28	Coke	2.73
Boeing	2.49	McD	2.32
Goldman Sachs	2.17	MMM	3.71
Caterpillar	3.00	Merck	2.96
Cisco Systems	3.11	MSFT	1.11
Chevron	4.73	Pfizer	4.24
Dow chemical	NA	Proctor & Gamble	2.36
Walt Disney	1.27	Nike	0.96
Walgreen	3.53	Travelers	2.45
Home Depot	2.22	United Health	1.43
Visa	0.57	United Tech	1.96
IBM	4.32	Verizon	NA
Intel	2.05	Wall Mart	1.84
Johnson^2	2.53	Exxon Mobil	5.81

12. Suppose that X and Y are jointly lognormally distributed, i.e.,

$$\begin{pmatrix} \log X \\ \log Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix} \right).$$

(a) Then show that

$$\frac{E(X)}{SD(X)} = \frac{1}{\sqrt{\exp\left(\sigma_X^2\right) - 1}}$$

$$\frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}} = \frac{\exp\left(\sigma_{XY}\right) - 1}{\sqrt{\left[\exp\left(\sigma_X^2\right) - 1\right]\left[\exp\left(\sigma_Y^2\right) - 1\right]}}$$

(b) Show also that

$$\log E(Y|X) = a + \beta \log X, \quad \beta = \frac{\sigma_{XY}}{\sigma_X^2}, \quad a = \mu_Y - \beta\mu_X + \frac{1}{2}\left(\sigma_Y^2 - \frac{\sigma_{YX}^2}{\sigma_X^2}\right)$$

$$\log(E(XY)) = \mu_X + \mu_Y + \frac{1}{2}\left(\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}\right) \neq E\left(\log(XY)\right) = \mu_X + \mu_Y.$$

Suppose that $\sigma_X^2 = k = \sigma_Y^2$ and $\sigma_{XY} = k\rho$, where $k = 1, 2, \ldots$

13. Use the FF market return R_{mt} and risk free rate R_{ft} daily, weekly and monthly frequency. Let $X_t = (R_{mt}, R_{f,t+1})^\intercal$ and estimate the Vector Autoregression

$$X_t = a + AX_{t-1} + \varepsilon_t.$$

Define the k-period ahead forecast based for $k = 1, 4, 5$.

6 Volatility

1. Using daily stock price data, calculate the daily gross return, the over night gross return, and the intraday gross return:

$$\mathcal{R}(t) = \frac{P_C(t)}{P_C(t-1)}, \quad \mathcal{R}_O(t) = \frac{P_O(t)}{P_C(t-1)}, \quad \mathcal{R}_I(t) = \frac{P_C(t)}{P_O(t)}.$$

   (a) Calculate the sample mean and variance of these three quantities and discuss your results with relation to the calendar time/trading time issue.

2. Suppose that stock prices P follows a geometric Brownian motion

$$d\log P(t) = \mu dt + \sigma dB(t),$$

where B is standard Brownian motion. Suppose that we observe transaction price $P(t_j)$ at time $t_j \in [0,1]$, $j = 0, 1, \ldots, n$, and let $r_{t_j} = \log P(t_j) - \log P(t_{j-1})$.

   (a) What is the distribution of r_{t_j}?

   (b) What is the joint distribution of $r_{t_1}, \ldots, r_{t_n}$? Hence write down the log likelihood for $\{r_{t_1}, \ldots, r_{t_n}\}$ and obtain the Maximum Likelihood Estimators of μ, σ^2

   (c) Define the realized volatility (RV) for the interval

$$\widehat{\sigma}^2_{[0,1]} = \sum_{j=1}^{n} r^2_{t_j}.$$

   Derive its Mean Squared Error.

3. Suppose that we observe transaction price P_{t_j} at time $t_j \in [0,1]$, $j = 0, 1, \ldots, n$, where t_j are ordered realizations from a uniform on $[0,1]$. Suppose that P_{t_j} is uniform on $[\theta_0, \theta_1]$. Consider the high/low volatility estimators

$$V^{HL} = \frac{P_H - P_L}{P_L}$$

$$V^P = \frac{(\log P_H - \log P_L)^2}{4\log 2},$$

where $P_H = \max_{0\le j\le n} P_{t_j}$ and $P_L = \min_{0\le j\le n} P_{t_j}$. In this case what do V^{HL} and V^P estimate and how does this relate to the volatility of P as estimated by the realized volatility calculated as

$$\sum_{j=1}^{n} \left(P_{t_j} - P_{t_{j-1}}\right)^2$$

4. Compute V^{HL}, V^P, and V^{RS} for daily S&P500 data. Also calculate the sample cross auto-covariance function between these volatility estimators and the intraday return measures (or rather the log, $\log \mathcal{R}_I(t)$ version of them) defined above and comment on the results.

5. Let $x_t = (\log \mathcal{R}_I(t), V_t^P)^\intercal$ and estimate the vector autoregression

$$x_t = a_0 + Ax_{t-1} + \varepsilon_t$$

using daily data on the S&P500. Hence, estimate for $k = 1, 2$

$$E\left(\log \mathcal{R}_I(t+k) | x_t, x_{t-1}, \ldots\right)$$

and plot the estimated conditional expectation.

6. Suppose that $x_t = (\mathcal{R}_I(t), V_t^P)^\intercal$ is conditionally lognormally distributed with

$$y_t = \begin{pmatrix} \log \mathcal{R}_I(t) \\ \log V_t^P \end{pmatrix}$$

satisfying

$$y_t = a_0 + Ay_{t-1} + \varepsilon_t,$$

where $\varepsilon_t \sim N(0, \Sigma)$.

   (a) Now estimate

   $$E\left(\mathcal{R}_I(t+1) | x_t, x_{t-1}, \ldots\right)$$

   and plot the estimated conditional expectation. Use the expressions given in exercise x above.

7. Suppose that y follows a GARCH(1,1) process

$$y_t = \sigma_t \varepsilon_t$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma y_{t-1}^2,$$

where ε_t is i.i.d. mean zero and variance one with some distribution for which $E(|\varepsilon_t|) = \mu_j(\varepsilon)$, $j = 3, 4, \ldots$.

(a) Show that

$$E(y_t^4) < \infty \Longleftrightarrow \gamma^2 < \frac{1}{\mu_4(\varepsilon)}$$

(b) Show that the kurtosis of y satisfies

$$\kappa_4(y) = \frac{\mu_4(\varepsilon) - 3 + 2\mu_4(\varepsilon)\gamma^2}{(1 - \mu_4(\varepsilon)\gamma^2)}.$$

8. Consider the IGARCH(1,1) process,

$$y_t = \sigma_t \varepsilon_t$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + (1 - \beta) y_{t-1}^2.$$

This process is not weakly stationary. The differenced process

$$\sigma_t^2 - \sigma_{t-1}^2 = \omega + (1 - \beta)(\varepsilon_{t-1}^2 - 1)\sigma_{t-1}^2$$

has mean ω for all t (given starting values) as does

$$y_t^2 - y_{t-1}^2 = \sigma_t^2 - \sigma_{t-1}^2 + (\varepsilon_t^2 - 1)\sigma_t^2 - (\varepsilon_{t-1}^2 - 1)\sigma_{t-1}^2.$$

Recall that linear nonstationary processes, like unit root processes, can be made stationary by differencing.

(a) Is this process y_t^2 difference stationary under some conditions?

9. Suppose that

$$y_t = \sigma_t \varepsilon_t$$

$$\sigma_t^2 = \omega + \gamma y_{t-1}^2,$$

where ε_t is i.i.d. with mean zero and variance one.

(a) Derive the autocovariance function of $x_t = y_t^2$. Use this to suggest a method for estimating ω, γ.

(b) Derive the autocovariance function of $x_t = y_t^4$.

(c) Estimate the sample autocovariance of powers of the daily stock return data you obtained in exercise 1 and comment on the results.

10. Consider the GARCH (1,2) model:

$$r_t = h_t^{1/2} \eta_t$$

$$h_t = \omega + \beta h_{t-1} + \gamma_1 r_{t-1}^2 + \gamma_2 r_{t-2}^2$$

where h_t is the conditional variance of time t returns and η_t is a mean zero and unit variance series.

(a) Explain the restrictions on the parameters of the GARCH(1,2) model required to ensure that the long-run unconditional variance exists,

(b) Describe the unconditional variance in terms of these parameters,

(c) Discuss how the values of the parameters affect the persistence of the response of dynamic volatility to a return shock.

(d) Suppose that η_t is standard normal. What is the 99% Value at Risk for returns?

11. Suppose that

$$y_t = \varepsilon_t \sigma_t$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma y_{t-1}^2,$$

where ε_t are i.i.d. standard normal.

(a) What is $E(y_t^2 | y_{t-1}, y_{t-2}, \ldots)$?

(b) What is $E(y_t^2 | y_{t-2}, y_{t-3}, \ldots)$?

(c) What is $E(y_t^2|y_{t-j})$? You may calculate this by simulation methods. Intuitively expect that it is a quadratic function but this is hard to prove. If $\beta = 0$ (ARCH(1) case) we have

$$\begin{aligned}
E\left(y_t^2|y_{t-1}\right) &= \omega + \gamma y_{t-1}^2 \\
E\left(y_t^2|y_{t-2}\right) &= \omega + \gamma E(y_{t-1}^2|y_{t-2}) = \omega + \gamma(\omega + \gamma y_{t-2}^2) \\
E\left(y_t^2|y_{t-j}\right) &= \omega \sum_{l=0}^{j-1} \gamma^l + \gamma^j y_{t-j}^2
\end{aligned}$$

for all j. In this case the univariate regression functions are all quadratic with different coefficients.

7 Continuous Time Processes

1. Suppose that the stock price $P_t = B_t^2$, where B_t is standard Brownian motion.

   (a) Is this process a martingale?

   (b) Calculate $E(P_t|\{P_s,\, s < t\})$.

2. Show that for B_t standard Brownian motion we have

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t \sim \frac{t}{2}(\chi_1^2 - 1).$$

3. Suppose that $Y_t\,, t \geq 0$ and $Z_t, t \geq 0$ are standard Brownian motions independent of each other. Let

$$X_t = Y_t^2 + Z_t^2.$$

   (a) Show that

$$E_t\left(X_{t+h} - X_t\right) = 2h$$

   (b) Show that

$$E_t\left[\left(X_{t+h} - X_t\right)^2\right] = 8h^2 + 4hX_t.$$

4. Suppose that $B_t, t \geq 0$ is standard Brownian motion and let $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a strictly increasing function and let

$$Y_t = B_{g_t},$$

where $g_t = G(t)$.

(a) Calculate $E_t(Y_{t+h} - Y_t)$ and $E_t[(Y_{t+h} - Y_t)^2]$.

(b) Consider the process $X_t = \int_0^t \sigma^2(s)dB_s$, where $\sigma^2(\cdot)$ is a deterministic function of time. Let $G(t) = \int_0^t \sigma^2(s)ds$. Therefore show that $X_t = B_{g_t}$.

5. Consider the common discrete time model

$$z_{t+1} = \mu(z_t) + \sigma(z_t)\varepsilon_{t+1}$$

with ε_t i.i.d. mean zero and variance and $\mu(\cdot), \sigma(\cdot)$ are functions of a scalar state variable z_t. Show that this process cannot be in the Affine class unless $\mu(\cdot), \sigma(\cdot)$ are both linear and ε_{t+1} is Gaussian.

6. A discrete time Markov process X_t is one for which the distribution of X_t given the past depends only on the most recent past, that is

$$\Pr(X_t \leq x|\mathcal{F}_{t-1}) = \Pr(X_t \leq x|X_{t-1}).$$

Say whether the following processes are Markov

(a) $X_t = \mu + \rho X_{t-1} + \varepsilon_t$, where ε_t is i.i.d. $N(0, 1)$

(b) $X_t = \mu + \rho X_{t-2} + \varepsilon_t$, where ε_t is i.i.d. $N(0, 1)$

(c) $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$, where ε_t is i.i.d. $N(0, 1)$

(d) $X_t = \sigma_t\varepsilon_t$, where ε_t is i.i.d. $N(0, 1)$ and $\sigma_t^2 = \omega + \gamma X_{t-1}^2$

(e) $X_t = \sigma_t\varepsilon_t$, where ε_t is i.i.d. $N(0, 1)$ and $\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \gamma X_{t-1}^2$

(f) $X_t = 1$ if $X_{t-1} = X_{t-2}$ and $X_t = -1$ if $X_{t-1} \neq X_{t-2}$, where $X_1, X_2 = \pm 1$ with equal probability.

8 Extreme Values

1. Suppose that X is standard Cauchy with density and c.d.f.

$$f(x) = \frac{1}{\pi}\frac{1}{1 + x^2}$$

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x).$$

Let $Y = a + bX$.

(a) What is the density and c.d.f. of Y.

(b) Let $M_n = \min_{1 \le i \le n} X_i$. Then derive the limiting distribution of M_n.

(c) Let $M_n = \min_{1 \le i \le n} Y_i$. Then derive the limiting distribution of M_n.

(d) Derive the 99% Value at Risk for Y.

2. Suppose that X is *Lomax* distributed with survivor function and density function:

$$1 - F(x|\alpha, \lambda) = \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \quad x \ge 0$$

$$f(x|\alpha, \lambda) = \frac{\alpha}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-\alpha-1}, \quad x \ge 0$$

for any $\alpha, \lambda > 0$.

(a) Derive the quantile function of X and hence define the 99% Value at Risk.

(b) Show that $E(X^k) < \infty$ if and only if $\alpha > k$ and specifically

$$M = E(X) = \frac{\lambda}{\alpha - 1}, \quad \alpha > 1$$

$$V = \text{var}(X) = \frac{\alpha\lambda^2}{(\alpha - 2)(\lambda - 1)^2}, \quad \alpha > 2.$$

(c) Show that

$$\alpha = 1 + \frac{\lambda}{M}, \quad \lambda = M(\alpha - 1)$$

$$V = \frac{\lambda^2\left(1 + \frac{\lambda}{M}\right)}{\left(\frac{\lambda}{M} - 1\right)(\lambda - 1)^2}$$

and hence that λ solves the cubic equation

$$(1 - V)\lambda^3 + (M + 2V + MV)\lambda^2 - (2M + 1)V\lambda + MV = 0.$$

Hence suggest a Method of Moments estimator for λ and hence α when you have a sample $X_1, \ldots, X_n$ drawn from this population.

(d) The skewness of X satisfies

$$\kappa_3^2 = \frac{4(1 + \alpha)^2}{(\alpha - 3)^2}\frac{\alpha - 2}{\alpha}$$

Hence show that α satisfies the cubic equation

$$\alpha^3(\kappa_3^2 - 4) - \alpha^2 6\kappa_3^2 + \alpha(9\kappa_3^2 + 12) + 8 = 0.$$

Hence suggest a Method of Moments estimator for λ and hence α when you have a sample $X_1, \ldots, X_n$ drawn from this population.

(e) Write down the log likelihood function and derive an expression for the MLE of α, λ.

(f) Apply this to daily gross stock return data, i.e., estimate α, λ by either MoM or ML. Why is this a bad model for stock returns? $f(0) = \alpha/\lambda$

(g) For x large we have

$$\log(1 - F(x|\alpha, \lambda)) = -\alpha \log\left(1 + \frac{x}{\lambda}\right) \simeq -\alpha \log(x) - \alpha \log \lambda$$

You may estimate α by the log rank regression.

3. Pensions regulators emphasize the solvency probability for some portfolio with random return $w^\intercal X$, $\Pr(w^\intercal X > s)$, where s is some solvency threshold. Suppose that one must achieve at least 67% solvency probability. Suppose that $X \sim N(\mu, \Sigma)$. What is the optimal (in terms of mean return) unit cost portfolio weighting vector w subject to the restriction that the solvency probability must be greater than 0.67?

4. Suppose that (X_t, Y_t), $t = 1, \ldots, T$ are two demeaned and rescaled stock return series. We consider two transformations of the series. First, the marginal empirical distribution transform, whereby

$$\widetilde{X}_t = \widehat{F}_X(X_t), \widetilde{Y}_t = \widehat{F}_Y(Y_t),$$

where $\widehat{F}_X, \widehat{F}_Y$ are the empirical c.d.f.'s. Second, the polar coordinates transformation

$$\rho_t = \sqrt{X_t^2 + Y_t^2}, \quad \theta_t = a\tan 2(Y_t/X_t).$$

If Y, X are iid standard normal, then θ_t is uniform on $[-\pi, \pi]$ and ρ_t has the Rayleigh distribution on $\mathbb{R}_+$, i.e., $f_R(r) = r\exp\left(-\frac{1}{2}r^2\right)$. Furthermore, $\widetilde{X}_t$ and $\widetilde{Y}_t$ have marginal distribution $U[0, 1]$. For a pair of stock return series of your choosing investigate the properties of these two transformed series.